Geometry, Measurement and Statistics 6

Implementation Guide

Written and developed by

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www.oxfordprimary.co.uk/numicon

About Numicon

Numicon is a distinctive multi-sensory approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Numicon was founded in the daily experience of intelligent children having real difficulty with maths, the frequent underestimation of the complexity of the ideas that young children are asked to face when doing maths and recognition of the importance of maths to them and to society as a whole.

Numicon aims to facilitate children's understanding and enjoyment of maths by using structured imagery that plays to children's strong sense of pattern. This is done through research-based, multi-sensory teaching activities.

Numicon takes into account the complexity of abstract number ideas and seeks to foster the self-belief necessary to achieve in the face of challenge or difficulty.

Through the combination of communicating mathematically (being active, talking and illustrating), exploring relationships and generalizing, children are given the support to structure their experiences: a vital skill for both their mathematical and their overall development.

A multi-sensory approach, particularly one that makes use of structured imagery, provides learners with the opportunity to play to their strengths, thereby releasing their potential to enjoy, understand and achieve in maths. By watching and listening to what children do and say, this enjoyment in achievement is also shared by teachers and parents.

Numicon strives to support teachers' subject knowledge and pedagogy by providing teaching materials, Professional Development and ongoing support that will help develop a better understanding of how to encourage all learners in the vital early stages of their own mathematical journey.

Contents

Welcome to Geometry, Measurement and Statistics 6

Before you start teaching, we recommend you take some time to familiarize yourself with the Numicon starter apparatus pack C, the teaching resources and the pupil materials to help you and the children get the most out of using Numicon.

Use this Implementation Guide:

- to find out more about what Numicon is
- to find out how using Numicon might affect your teaching of geometry, measurement and statistics
- to learn about the key mathematical ideas children face in the Geometry and Measurement activity groups.

You will find guidance on how to get the most out of teaching, planning and assessing using Numicon in the Numicon Planning and Assessment Support on www.oxfordowl.co.uk.

You will find more information, videos, ideas and free resources on the Numicon website: www.oxfordprimary.co.uk/numicon. Here, you can sign up to receive the latest Oxford Primary news by email.

What's in the Numicon starter apparatus pack C?

The following list of apparatus supports the teaching of Geometry, Measurement and Statistics 6. These resources should be used in conjunction with the focus and independent practice activities described in the activity groups.

Apparatus pack contents

- Numicon Shapes – box of 80 (× 2)
- Numicon Coloured Counters – bag of 200 (× 2)
- Numicon Baseboard Laminate – set of 3 (× 2)
- Number rods – large set (× 1)
- Numicon 1–100 cm Number Rod Track (× 3)
- Extra Numicon 10-shapes – bag of 10 (× 3)

The following apparatus is not specifically listed for use in Geometry, Measurement and Statistics 6 activities, but should be used freely and as needed to support and extend children's work.

- Numicon ¯12–12 Number Line (× 1)
- Numicon 0–100 Numeral Cards
- Numicon Spinner (× 4)
- Numicon 0–100 cm Number Line – set of 3 (× 2)
- Numicon 1 000 000 Display Frieze (× 1)
- Numicon 0–1·01 Decimal Number Line (× 1)
- Numicon 10s Number Line Laminate/s (× 4)
- Numicon Fraction Number Line Laminate (× 1)
- Numicon 1–100 Card Number Track (× 3)
- Numicon Feely Bag (× 2)
- Magnetic strip (× 1)

Numicon Shapes 1

These offer a tactile and visual illustration of number ideas, and may be used to support work on transformations and calculating in Geometry, Measurement and Statistics 6.

Numicon Coloured Counters 2

These red, blue, yellow and green Counters are useful for making patterns and arrangements, and for working on position, direction, movement, area, transformations and symmetry, on a Baseboard Laminate or grid.

Numicon Baseboard Laminate 3

This doubled-sided laminated square baseboard is an empty 100 square, scaled to take Numicon Shapes and Counters. The white side is used in many activities, providing a defined 'field of action' for work on position, direction, movement, transformations and area in particular. The orange side is a decimal baseboard laminate which can be used to help children explore decimal numbers by offering a possible representation of an expanded 1-shape. Through carefully scaffolded progression first introduced in the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, children can use Numicon Shapes on the baseboard to represent decimal parts of numbers.

Number rods 4

A set of ten coloured rods, 1 cm square in cross-section. The shortest is 1 cm in length, the longest 10 cm. These offer another structured illustration for number. Being centimetre-scaled, they can also be placed along the Numicon 1–100 cm Number Rod Track or 0–100 cm Number Line and are also useful for exploring volume and capacity in cubic centimetres and millilitres.

Numicon 1–100 cm Number Rod Track 5

The 1–100 cm Number Rod Track supports children's use of number rods and measuring work. The decade sections click together into a metre long track, and can also be separated into sections to form an array.

Numicon ¯12–12 Number Line 6

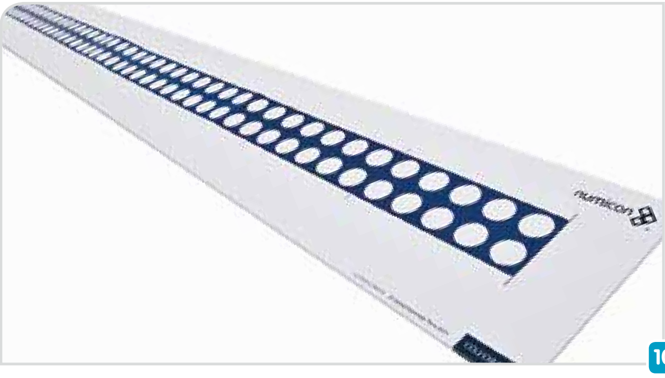
This number line shows negative and positive numerals. It provides a visual reference for counting and calculating with negative numbers when exploring temperature and can be displayed on the wall or given to children for use on their tables.

Numicon 0–100 Numeral Cards 7

The pack of 0–100 Numeral Cards may be used in focus activities, whole-class and independent practice activities and games to generate numbers for children to work with.

Numicon Spinner 8

Different overlays (provided as photocopy masters) can be placed on the Numicon Spinner to generate a variety of amounts, e.g. angles in degrees.



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Numicon 0–100 cm Number Line 9

The points on this number line are 1 cm apart and are labelled from 0 to 100. The number line is divided into decade sections, distinguished alternately in red and blue, to help children find the '10s' numbers that are such important signposts when they are looking for other numbers. It can also be used with number rods as an alternative to the 1–100 cm Number Rod Track.

Numicon 1 000 000 Display Frieze

This display frieze shows 100 blocks of 100 squares, each made up of 100 dots, helping children recognize 1 000 000 as a cube number. The sections can be arranged end-to-end horizontally or as an array, such as in a square of 10 × 10 blocks, helping children also recognize 1 000 000 as a square number. It provides a visual reference point for the scale of large numbers as well as supporting discussions about the application of square and cube numbers to area and volume.

Numicon 0–1·01 Decimal Number Line

The points on this number line represent thousandths, hundredths and tenths. Zero and one are labelled while the other points are left blank to encourage children to think about what each point shows, and children could record units of measure and decimals in tenths, hundredths or thousandths on this number line.

Numicon 10s Number Line Laminate 10

This laminated number line, scaled to take Numicon Shapes, shows Numicon 10-shapes laid horizontally end-to-end with points marked, but not labelled. The points can be labelled with multiples, fractions, decimals, percentages, negative numbers or units of measure, using a whiteboard pen, to support children with exploring number relationships and calculating.

Numicon Fraction Number Line Laminate

This laminated number line starts at zero and has fifty unlabelled points. The points can be labelled with any denomination of fraction children are working with. This provides a valuable tool for exploring equivalences and comparing fractions with different denominators.

Numicon 1–100 Card Number Track

This number track is divided into ten strips, numbered 1–10, 11–20, 21–30, and so on. The sections can be arranged end-to-end horizontally or as an array similar to a 100 square. It can be used to support number work in any geometry or measurement context.

Numicon Feely Bag 11

The Feely Bag can be used to generate numbers for use in problems and calculations involving measurement, e.g. by choosing from a selection of number rods, Numicon Shapes or coins in the Bag.

Magnetic strip

This self-adhesive magnetic strip can be cut into pieces and stuck onto Numicon Shapes or number rods so that they can be used on a magnetic whiteboard.

Available separately

Numicon Software for the Interactive Whiteboard 12

This rich interactive tool is designed for use with the whole class to introduce key mathematical ideas. It includes: number

lines featuring the Numicon Shapes, the Numicon Pan Balance, Numicon Spinners, shapes, coins and much more.

Numicon Pan Balance 13

Using Numicon Shapes, number rods or other objects in this adjustable Pan Balance (which also features on the *Numicon Software for the Interactive Whiteboard*) enables children to see equivalent combinations. In particular this helps them to understand that the = symbol means 'is of equal value to' and avoid the misunderstanding that it is an instruction to do something. Children can easily see what is in the transparent pans, making the Pan Balance especially useful for comparing quantities as part of measurement work. It can also be used, in the same way as any bucket balance, to explore mass.

Individual sets of Numicon Shapes 1–10

Designed for multi-sensory, whole-class lessons where each child has their own set of Numicon Shapes and is encouraged to engage with them. In Geometry, Measurement and Statistics 6 children can use them for support in handling numbers as part of measurement work. They can also be used in conjunction with the *Numicon Software for the Interactive Whiteboard* to help teachers assess children's individual responses.

Other equipment recommended for the teaching of geometry and measurement

Other resources that are typically found in classrooms may be used to support work in the activity groups. Resources that are particularly useful for geometry and measurement activities are described in more detail here. Other more generally useful items, such as **imperial measuring equipment**, **base-ten apparatus**, **interlocking cubes**, **string**, **squared paper, graph paper, spreadsheet software** and so on, are highlighted in the '**Have ready**' sections of the focus and practice activities.

Pattern blocks 14

Pattern blocks are a side-length-matched set of shapes which children can use flexibly to explore and understand the parts and properties of shapes. They can be used to create patterns and pictures, as well as to investigate scale, transformations, tessellations, repeating sequences or symmetrical designs.

Geo strips and connectors 15

By connecting geo strips – flexible, punched strips of various lengths – children can make, alter and remake shapes for themselves, helping them to link the physical, variable properties of their models to concepts of space, line, angle and shape. Children may find geo strips useful when exploring the properties of 2D shapes in Geometry 1.

Geo boards and elastic bands 16

This equipment is needed for GMS Investigating 1. Stretching an elastic band around the pegs on a geo board is a quick and simple way for children to make and alter shapes, allowing them to explore a variety of geometric concepts, including angles, symmetry, area and perimeter, transformations and coordinates, among others. Children's ideas and results can also be recorded onto matching 'dot' paper, making the geo board a useful investigative tool.

A number of types of geo board are available, the most common of which has a square grid of pegs. Others have an 'isometric' arrangement, which can be used to illustrate equilateral triangles and a wide variety of 2D shapes and 2D representations of 3D shapes. Note that isometric 'dot' paper (photocopy master in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*) can be used for recording in this instance.

Geometric shapes 17

Geometry, Measurement and Statistics 6 places particular emphasis on using cubes and cuboids to explore ideas of solid volume and capacity.

Images and examples of a variety of geometric shapes should also be available for children to look at, handle and talk about, ensuring that they gain broad experience and maintain fluency in naming and describing the different types. Provide flat, 2D shapes ranging across both non-polygons (circles, semicircles and ovals, as well as, crescents, heart shapes, cloud shapes, shapes with rounded corners) and polygons (regular and irregular, including rectilinear shapes).

Similarly, provide solid, 3D shapes to illustrate both non-polyhedra – spheres, hemispheres, spheroids (rugby ball or 'squashed sphere' shapes), cones, cylinders, toruses (doughnut or tyre shapes) – and polyhedra, including the Platonic solids (the tetrahedron, cube, octahedron, dodecahedron and icosahedron) and a range of different prisms and pyramids.

Images and objects illustrating the use of shape in everyday life, in a variety of sizes, forms and contexts, can also be provided, to encourage children to expand their understanding and appreciation.

Programmable robots

A number of different programmable floor robots for classroom use are available. These require the user to enter a series of instructions, typically for turns and straight-line movements, which they then follow to complete a journey around the floor. They represent a practical and appealing context for demonstrating, thinking about and experimenting with position, direction and movement, angles and turns, and measurement, among other topics, as well as for problem solving and logical thinking more generally. They are also a means of encouraging children to begin making links between mathematics and ICT, and of approaching the idea of programming.

Geometry software

A variety of geometry software packages for schools are available. They give a practical context to illustrate and visualize turns, angles, triangles and other 2D shapes in a clear way. They are also a means of encouraging children to link mathematics and ICT, and to extend their knowledge of uses of ICT.

Items and materials for measuring; measuring instruments

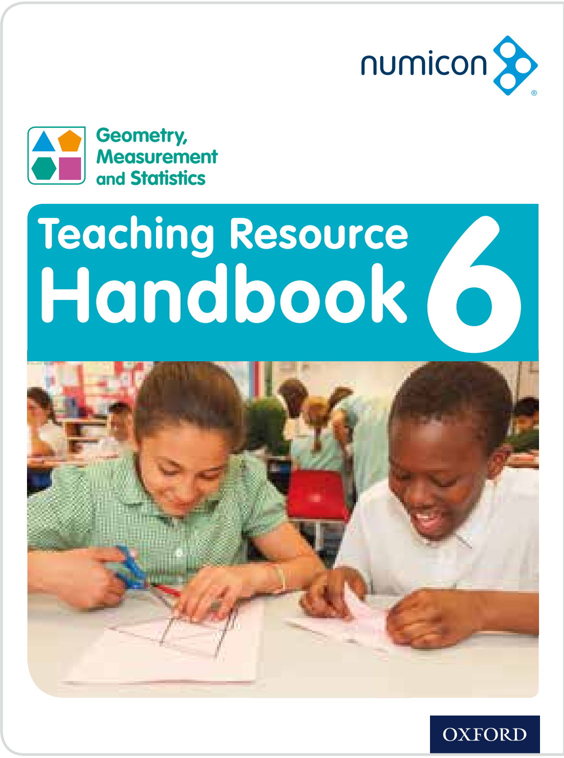
In Geometry, Measurement and Statistics 6, children use standard units of measurement for length, area, mass and capacity. They also work with units of volume and imperial units for length, mass and volume.

Children build on the work they did in Geometry, Measurement and Statistics 5 on area and perimeter. They build on their knowledge of areas of rectangles to find the areas of other quadrilaterals and triangles. This draws them into using algebraic notation to express the area of any triangle as $\frac{1}{2}bh$.

Children continue to record measurements in metric and imperial units and consider the relationship between the two. For exploring imperial units, it is useful to have a variety of measuring equipment marked in imperial units.

Many of the activities in Geometry, Measurement and Statistics 6 involve exploring angles using degrees as a measure of angle and turn. Children make use of protractors to look at individual angles, triangles and other 2D polygons. It is also useful to have board protractors to hand to practically demonstrate angles for the whole class.

In general, it can be useful to present children with measuring instruments which vary in form and appearance – in the design and materials used, as well as the interval marks and labels on the scales. This will help to reinforce children's understanding of imperial and metric units and the relationships between them; to see that, while the appearance of rulers, tape measures, metre sticks, trundle wheels and so forth varies, the centimetres they show are all the same length. If possible, provide a variety of examples and images of measuring instruments to help broaden children's experience.



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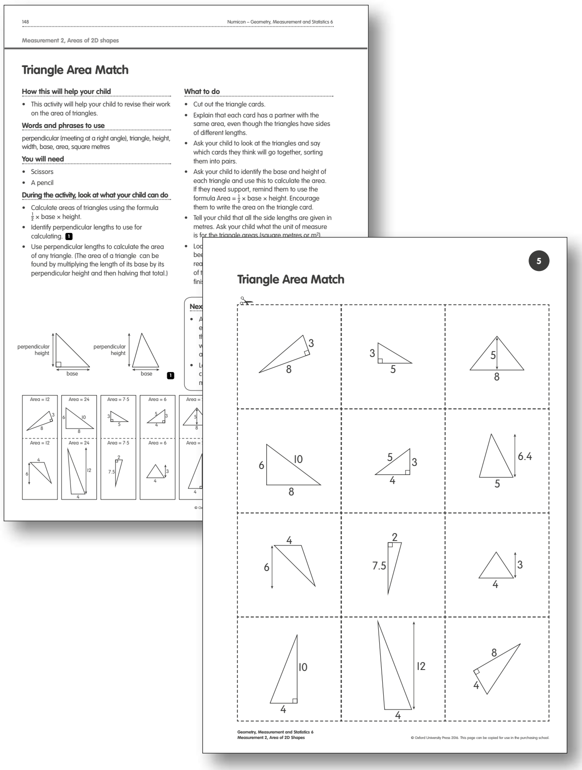


19

What's in the Numicon teaching resources?

Geometry, Measurement and Statistics 6 Teaching Resource Handbook 18

This contains 10 activity groups clearly set out and supported by illustrations. Each core activity group begins with the educational context, learning opportunities, assessment opportunities and important mathematical vocabulary.
To support teachers' assessing of children, there are notes on what to 'look and listen for' as children work on the activities, as well as suggestions for whole-class and independent practice. Photocopy masters supporting the activity groups are included at the back of the Teaching Resource Handbook.

Support for planning and assessment is included in the front of the handbook. There, you will find:

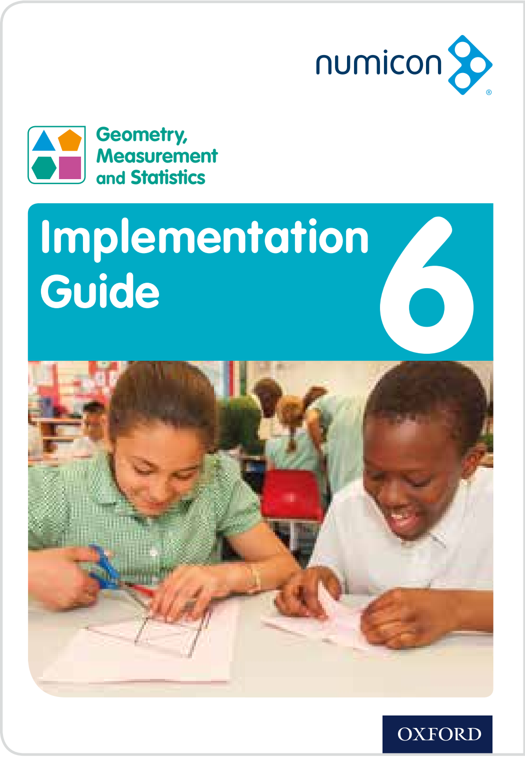
- information on how to use the Numicon teaching materials and the physical resources
- long- and medium-term planning charts that show the recommended progression through the activity groups
- an overview of the activity groups.

Geometry, Measurement and Statistics 6 Explore More Copymasters (provided in the Teaching Resource Handbook) 19

The Explore More Copymasters offer children the chance to practise and discuss mathematics at home with a parent or carer. An activity has been included for each activity group so that children have ongoing opportunities to practise their mathematics learning outside of the classroom.

Each activity is supported by information for the parent or carer on the mathematics that has been learned in class beforehand and the learning point of the activity itself. Guidance on how to complete the activity is included, as well as a suggestion about how to make the activity more challenging or how to develop the activity further in a real-life situation.

The Explore More Copymasters can be given to an adult or child as a photocopied resource.



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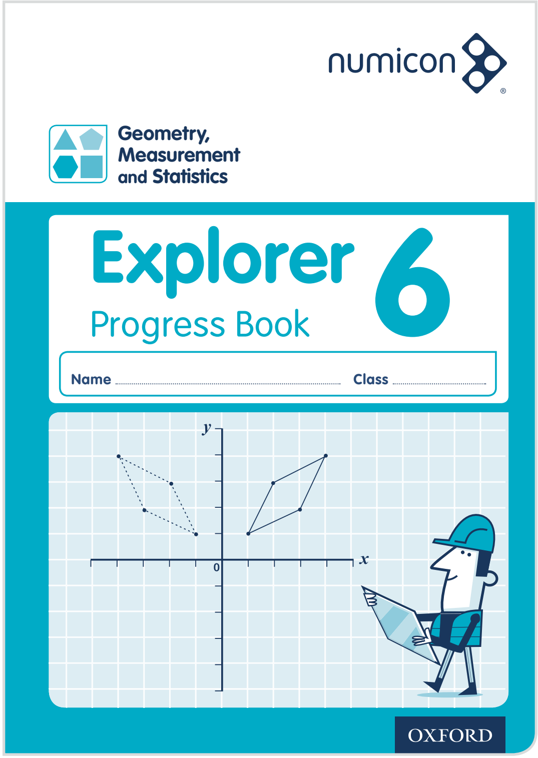


21

***Geometry, Measurement and Statistics 6 Implementation Guide* 20**

This introduces what Numicon is and how Numicon helps children meet the demands of learning mathematics. It also includes some practical advice about what to do when preparing to teach with Numicon and answers some key questions about how to use Numicon in practice. The chapters on 'Key mathematical ideas' provide useful explanations of the important concepts children will meet in the 10 activity groups of the Teaching Resource Handbook and how these ideas develop when teaching with Numicon.

The different sections of the Implementation Guide can be accessed as and when necessary to help you with your teaching.

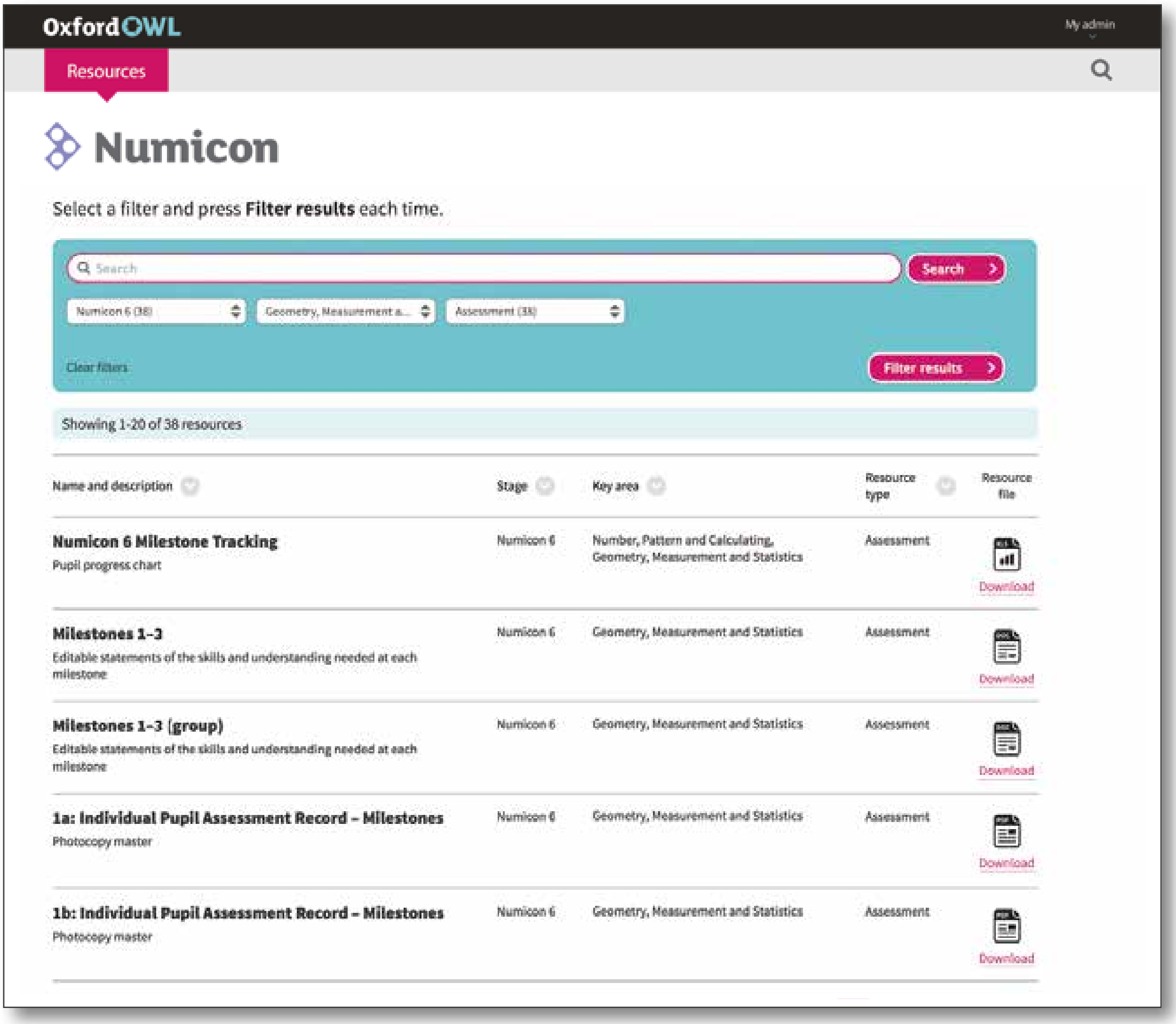
***Geometry, Measurement and Statistics 6 Explorer Progress Book* 21**

The Explorer Progress Book offers children the chance to try out the mathematics they have been learning in each activity group and teachers the chance to assess their understanding. It should be stressed that the challenges in the Explorer Progress Book are not tests. There are no pass or fail criteria; the challenges are simply designed to reveal how well children can use the mathematics they have been learning in new situations.

Being able to use mathematics in an unfamiliar situation is a significant indicator of children's understanding. Many of the tasks set mathematics in a new or different context and, where possible, the challenges are set in an 'open' way, inviting children to show how they can reason with the ideas involved rather than testing whether they have learned a routine solution to routine tasks.

As with children's classroom activity work, a range of apparatus should be freely available to children as they reason with the ideas in their Explorer Progress Book.

In addition, there is also scope for self-assessment. This can be used flexibly, or to summarize learning at the end of a block of work.



Numicon Planning and Assessment Support

The Numicon Planning and Assessment Support is available in the Teaching and Assessment Resources on the Oxford Owl website and contains a wealth of digital resources that will assist you with your planning and assessment needs.

Within the support you will find short videos introducing Geometry, Measurement and Statistics 6 and offering advice that will help you get started teaching with Numicon.

There is also an editable summary of each activity group, including the title and number of the group, the educational context, learning opportunities, assessment opportunities, and the mathematical words and terms to be used with children as they work on the activities. There are suggestions for how to use these activity group summaries in your planning, as well as editable short-, medium- and long-term planning frames. A teaching progression chart is included which shows the reccomended order for all the activity groups across Number, Pattern and Calculating 6 and Geometry, Measurment and Statistics 6.

There is a milestone tracking spreadsheet to support you in monitoring each child's progress throughout the year. Assessment grids are also available to record children's achievements in each activity group and during work in the Explorer Progress Book.

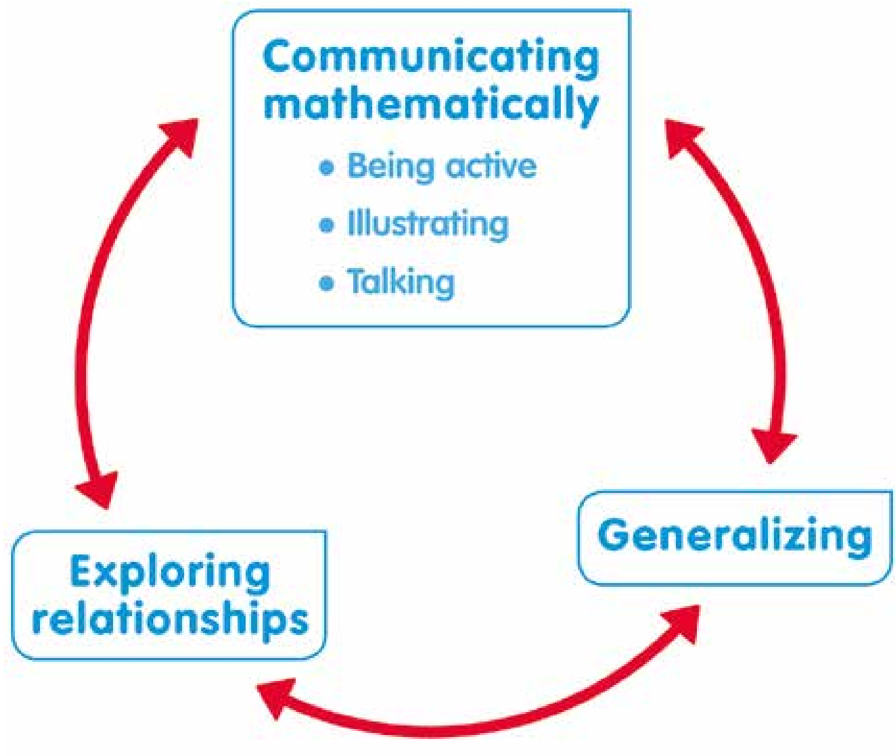
Charts that map Numicon to the English, Welsh, Scottish and Northern Irish curricula have been included in these resources.

Printable versions of all the photocopy masters are also included.

What is Numicon?

In order to illustrate how using Numicon in your teaching can help children learn mathematics, this section looks at:

- what Numicon is
- what doing mathematics demands of children
- how using Numicon helps children meet these unique demands.



What is Numicon?

Numicon is a distinctive approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Communicating mathematically

Doing mathematics involves communicating and thinking mathematically – these are two sides of the same coin. We think in the same ways that we communicate and communicate in the same ways that we think. As children learn to communicate mathematically, they learn to think mathematically. This involves them in the following:

Being active: Teaching and learning with Numicon requires children to be active. This does not only mean being physically active (e.g. fitting physical objects together, finding a number on a number line, drawing a shape), but is a requirement reflecting the understanding that mathematics itself is activity: mathematics is something children are learning to *do*.

What this means in practice is that it is always children themselves who are to do the mathematics. Telling children (or showing, or explaining) 'what to do' can encourage children to be passive. Numicon asks that children do the mathematics (that is both the activity and the thinking). In other words, children actively learn to do mathematics for themselves.

Illustrating: Doing mathematics (that is thinking and communicating mathematically) necessarily involves illustrating, because mathematics is about studying relationships between objects, actions and measures, and it is impossible to explore such relationships without some kind of spatial imagery being involved. Even when thinking about just two numbers, one of them 'comes after' the other one on a number line or is said to be 'bigger' or 'higher'; these relations all involve spatial imagery.

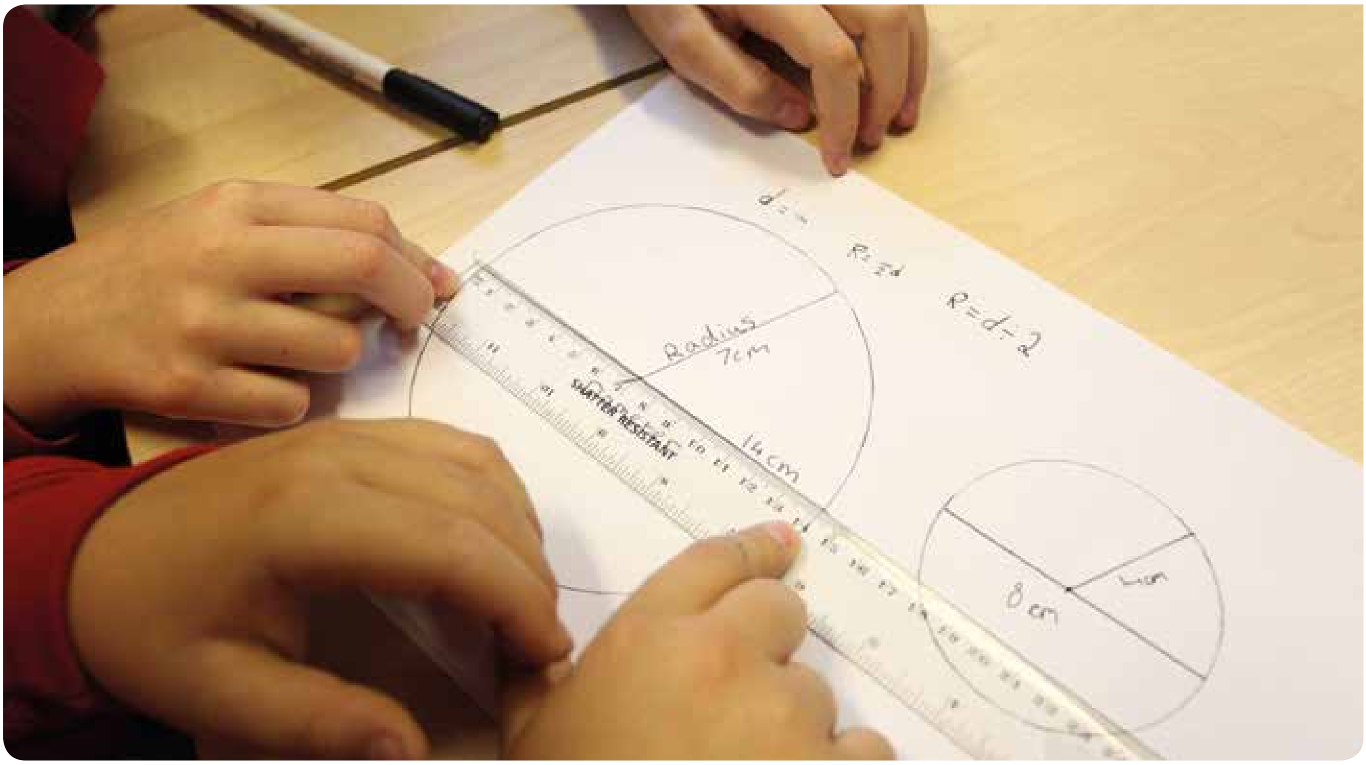
Numicon explicitly involves illustrating in every activity. This is how relationships in a wide variety of contexts are both explored and communicated.

Talking: Since doing mathematics involves communicating mathematically (both with others and with ourselves), doing mathematics also involves talking. Talking is an essential aspect of all Numicon activity.

Throughout Numicon, talking means dialogue in which points of view are exchanged between teachers and children, and between children and children. All partners in dialogue will be actively involved, not just passively hearing or waiting to speak. Such exchanges encourage reflective thinking as children learn to discuss different points of view, both with others and with themselves.

Exploring relationships (in a variety of contexts)

Doing mathematics involves **exploring relationships** (the structure) in any situation in order to develop some kind of control in that area. Relationships can be between amounts, between positions, between shapes, between things that are varying, between things that are constant, or between combinations of all of these things. Fully understanding the relationships in a situation makes it predictable, thus making it possible to find out and/or manage what is needed. The mathematical reasoning children are asked to do is their expression of relationships they perceive.



Numicon ensures that children explore relationships within a wide range of contexts so that they learn not only *how* to do mathematics, but *when* the mathematics they are learning is useful.

Generalizing

In *doing* mathematics, exploring relationships and looking for patterns in various situations lead to **generalizing**. It is generalizing that allows us to predict whole ranges of new situations.

Numbers are generalizations that we all use to make predictions when calculating. For example, the '6', '2' and '8' in the number sentence '6 + 2 = 8' are generalizations; 6 of anything and 2 of anything, will together always make 8 things, whatever they are.

'The angles of a triangle add up to 180°' is a generalization that is often used when doing geometry. 'The area of a circle is πr^2' is another that is used when measuring.

In each of these cases, noticing patterns in relationships allows us to generalize about an infinite number of other, similar situations.

We make and use generalizations continually as we do mathematics, but mathematical thinking and communicating will appear to have an abstract character for children if they are not involved in doing the generalizing for themselves.

Communicating mathematically, **exploring relationships** and **generalizing** all come together when *doing* mathematics.

What doing mathematics demands of children

In learning to do mathematics at the age they are asked to, children face a unique challenge in their school curriculum: thinking and communicating about abstract objects.

Pure numbers, such as 6 and 254, are abstract objects and in no other subject are children asked to work with and reason about such things so early. It is not surprising that children can hesitate, be puzzled, get stuck or take time to work things out for themselves.

Most mathematicians will say that doing mathematics is about looking for patterns in situations. It is, but there is more to it than that. When you find a pattern, it means you have noticed something regular, something that always seems to be happening, and this means that any pattern you see is a generalization of yours. Mathematics deals almost entirely in generalizations.

Numbers are generalizations; they are also abstract objects. Very early on in mathematics, young children are asked to do things with lots of abstract objects, like those we call '3', and 'ten'. Not '3 pens', or 'ten sweets', or '3 friends'. Just '3', or, even more curiously, the two-digit '10' by itself. Notice how quickly children are asked to add and subtract these abstract objects to and from each other. There are high expectations of every child from the very beginning in mathematics.

The central problem: communicating with, and about, abstract objects

How is it possible to communicate about abstract objects? And, since thinking is communicating with ourselves, how is it possible to *think* about abstract objects?

Notice that 'abstract' does not mean 'imaginary'. We can easily picture imaginary things, such as unicorns and beings from outer space, but abstract things are different. Abstract things include qualities like 'truth', 'intelligence', 'common sense', as well as generalizations such as '6 of anything'. The problem is, as soon as you try to picture '6 of anything' you find you are imagining '6 of something'.

The answer, as Bruner observed, is that we think about abstract objects with symbols; in the case of numbers, we do it with numerals.

The important thing about symbols is that they do not attempt to show literally what is being talked about; they are simply arbitrary, conventional marks on a page (or spoken words). When what we are communicating about with these symbols is abstract, that absence of a picture is inevitable; how could we possibly picture something that is abstract?

It is easy to picture '3 pens' or 'ten friends', but what might the abstract '3' look like? Or, how about the curious two-digit '10'? Since numerals do not look like the abstract things they 'stand for', how are children to learn to interpret such arbitrary symbols in their thinking and communicating? Doing mathematics – thinking and communicating about abstract things with symbols – is certainly not easy for young children.

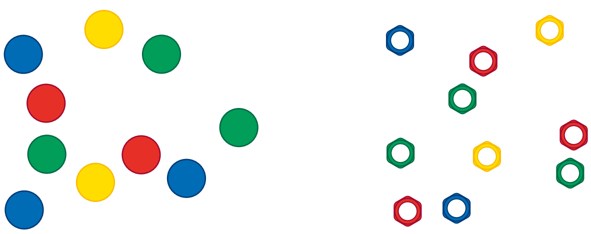
How does Numicon help?

Essentially, Numicon does two things. Firstly, Numicon acknowledges that in order to understand what numbers are, children have to generalize. Secondly, Numicon follows Bruner's advice in using children's actions and imagery to prepare for their use of mathematical symbols in their thinking and communicating. In practice, these two things are done at the same time.

In Bruner's terms, *enactive* and *iconic* representations (action and imagery) are used to inform children's interpretation of the *symbolic* representation (e.g. numerals) that is necessary for communicating their pattern spotting (generalizing). To aid children's necessary generalizing, strong focus is placed upon the use of structured materials.

Generalizing and reasoning – an early years example of teaching with Numicon

Initially, in common with most teaching, Numicon involves a wide variety of everyday objects (such as beads, cubes, pegs and counters, see Fig. 1) in order to help children develop their counting, before then introducing the challenges of calculating.



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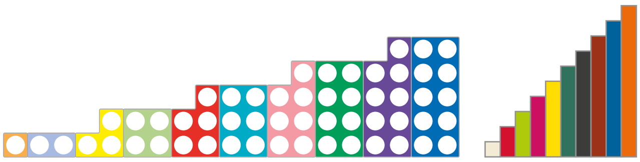
Importantly, Numicon also introduces sets of structured materials in which individual pieces have *regular* physical relationships with each other, e.g. Numicon Shapes and number rods, see Fig. 2. Children explore the physical relationships within these structured materials by, e.g. ordering pieces, comparing them, combining them physically to make others.



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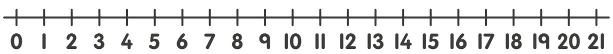
So, as children work with loose collections (beads, cubes) and with structured materials (Numicon Shapes and number rods), they are able to combine being active with physical objects and images as illustrations in their talking and thinking about numbers of things.

Numerals are introduced in association with loose collections of objects and with structured materials as children work; thus, symbolic representation appears alongside children's ongoing action and imagery with physical objects.

Number lines (e.g. Fig. 3) are introduced to further illustrate the ordering that is evident within the structured materials, and to reinforce the associated ordering of numerals.



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Importantly, loose collections of objects are arranged into the regular patterns of the Numicon Shapes. Individual number rods are found to be equivalent in length to different multiples of the smallest 'unit' rod (e.g. Fig. 4).

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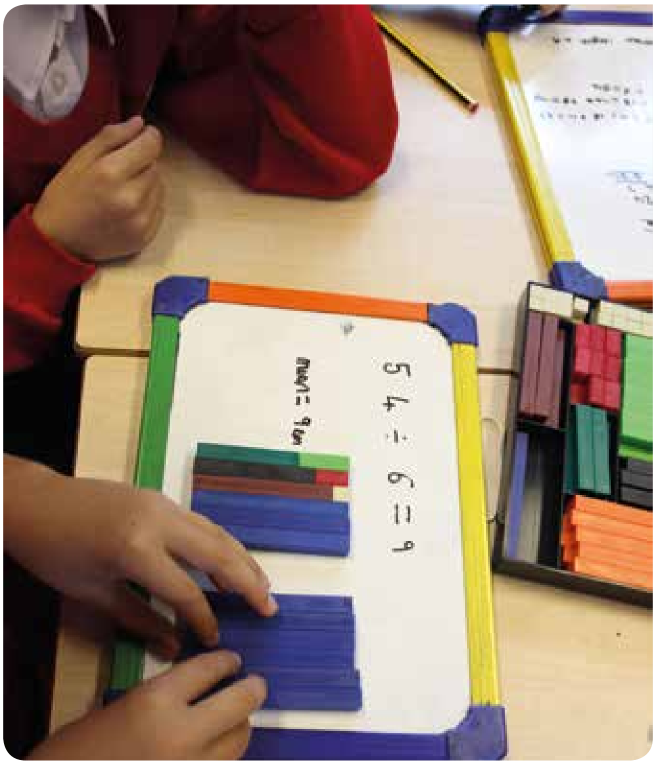
Through doing these activities, children learn that *any* collection of loose objects can be arranged into Numicon Shape patterns that can then be 'read'. Children learn to 'see' how many objects are in a collection without counting; numbers of things begin to 'take shape' visually, in organized ways.

It is now possible to prepare children for further generalizing about numbers through exploring relationships between numbers of things. Children's mathematical thinking and communicating continue to develop through their being active with the objects they are given, and in illustrating their talking about what they see and do.

Children generalize that any collection of loose objects can be arranged into Numicon Shape patterns. They also generalize that any number of 'unit' number rods may be exchanged for (will be 'as long as') one or more of the larger number rods. Thus *any* number of loose objects can be converted into (is equivalent to) one or more Numicon Shapes or number rods.

In realizing that any collection of loose objects can be arranged into Numicon Shape patterns, and that any number of unit



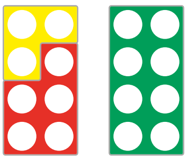
cubes is equivalent to one or more number rods, children become able to 'see the general' in any particular illustration; Numicon Shapes and number rods can be used to illustrate, in an organized way, any numbers of any kinds of things.

Numicon Shapes and number rods themselves may now be used to explore and to communicate about number relationships in general. They have become communication mediators in discussions about numbers and their relationships.

As a particular example, the Numicon Shape that has three holes fits together physically with the Shape that has five holes. The result 'forms the same shape as' (is equivalent to) the Shape that has eight holes, see Fig. 5.

Similarly, the number rod worth three units, combined end-to-end with the rod worth five units, are together as long as the rod worth eight units, see Fig. 6.

When laid end-to-end along a number line or number track, the '3 rod' and the '5 rod' together reach the position marked '8' on the line.



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From these actions, and with these illustrations, children are able to generalize that three *anythings* and five *anythings* together will *always* make eight things.

This generalization can be expressed using the numeric and verbal symbols: 3 and 5 together are equivalent to 8.

Later on, using further actions and further illustrations, children become able to interpret and use the further symbols '+' and '=' to express their generalization:

$$3 + 5 = 8$$

Importantly, at this stage, children will have begun to use number words (one, two, three) as *nouns* instead of as adjectives (two sweets, three pencils) in their talking.

With their use of number words as nouns, numbers as *abstract objects* have now appeared in children's mathematical thinking and communicating, referred to with *symbols*.

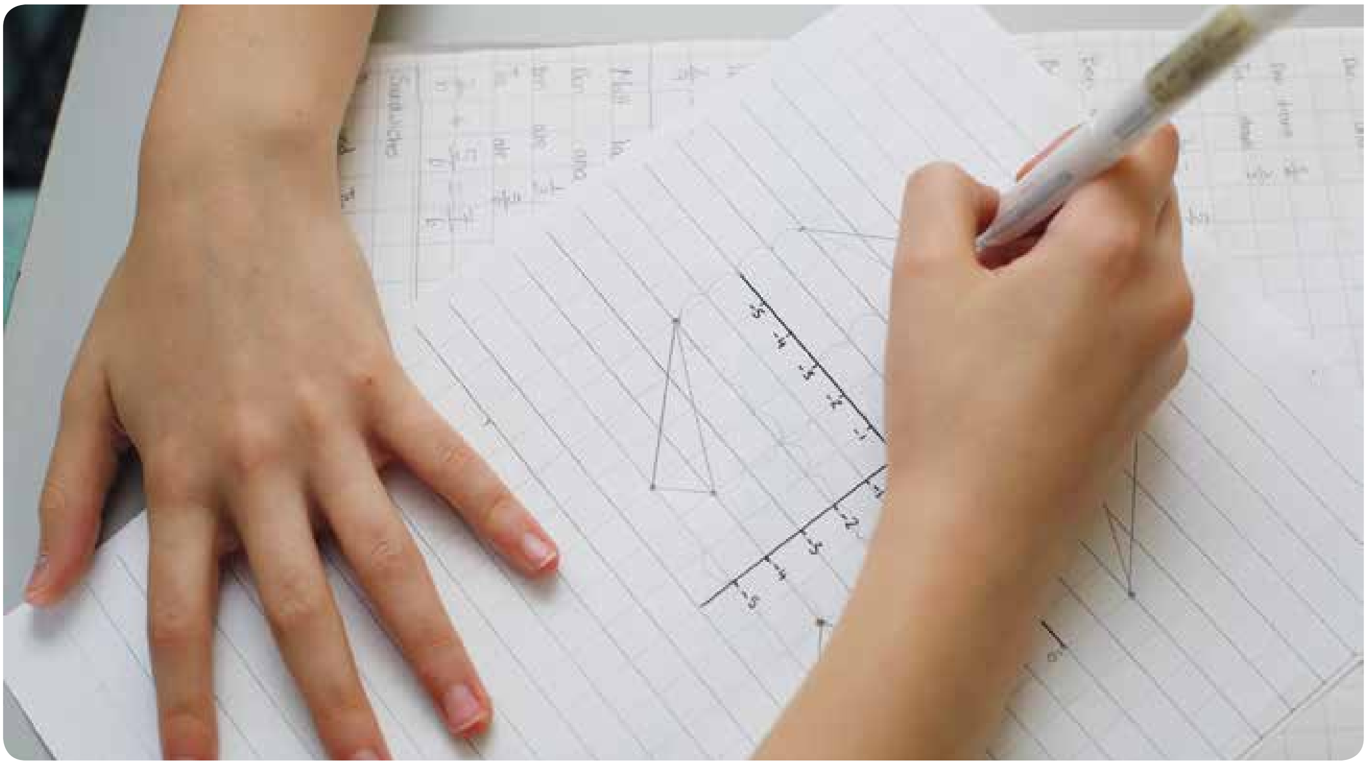
Such generalizing and use of symbols can now be developed further. If 'three of *anything*' and 'five of *anything*' together always make 'eight *things*', then:

3 tens + 5 tens	=	8 tens
3 hundreds + 5 hundreds	=	8 hundreds
3 millions + 5 millions	=	8 millions

or

30 + 50	=	80
300 + 500	=	800
3 000 000 + 5 000 000	=	8 000 000

This is the power of generalizations, and of the symbolic notation that children can by this stage use to think and communicate mathematically about them. The ability to generalize and predict is the true power of mathematics. Later on children will learn to call it 'algebra' and their arithmetic skills will be enhanced once they have grasped this central concept.



Progressing from such early beginnings

The previous example also illustrates how Numicon supports the teaching of children's later mathematics.

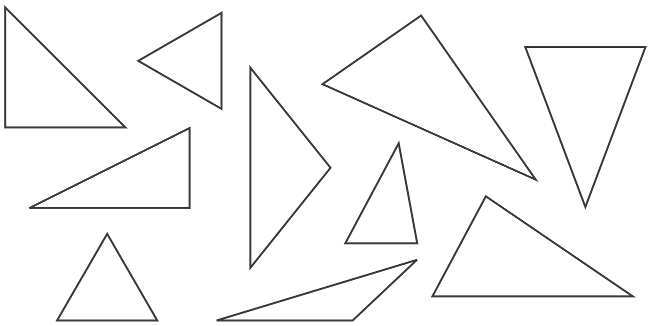
In a wide variety of contexts, children are offered opportunities to be active and to illustrate their talking about any (and all) relationships they are exploring between both physical and abstract objects. They do this in order that they may generalize and, as a consequence, think and communicate about their generalizations with the conventional symbols of mathematics: in other words, that they may *do* mathematics.

Of course, there is much more beyond this first example. Children have many other relationships to explore, further generalizing to do, and many more conventional symbols to learn to interpret and use. However, wherever it is used, the approach is essentially the same. *Enactive* and *iconic* representation are used to inform children's interpretation of the *symbolic* representation that is necessary for thinking and communicating about their constantly developing pattern spotting (generalizing).

The going gets harder as children progress in their number work: place value, ratios, fractions, decimals and negative numbers will all challenge children's capacity to generalize onwards from these early beginnings. However, such advanced work is equally possible with the same approach of actions and illustrations building towards generalizing, and a consequent use of symbols to think and communicate about the generalized, abstract mathematical objects created. This is how the symbolism of mathematics becomes meaningful.

In their work on geometry, the generalizations that children reach in their mathematical thinking and communicating happen more gradually as they progress towards being able to reason about invented abstract objects such as 'any triangle' and (later) 'any polygon'.

It is impossible to draw the abstract object 'any triangle' in the same way that it is impossible to imagine 'six of anything'. As soon as you draw a triangle, however you have chosen to draw it, you have drawn a particular one. It does not matter whether you draw one that is right-angled, isosceles, equilateral, or scalene, what you have drawn is not a general triangle – it is a particular one, e.g. Fig. 7.



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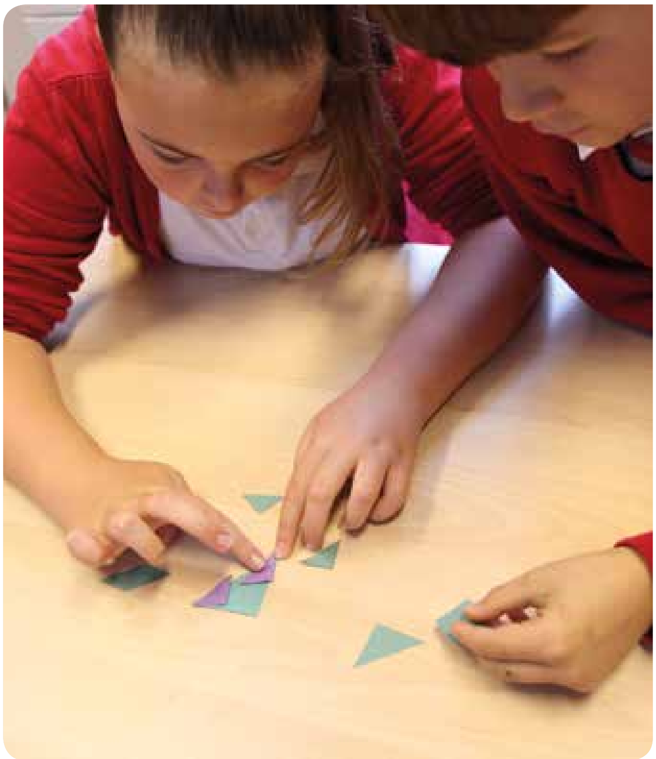
However, as with generalizing about numbers, in *doing* geometry, much exploring of relationships with action and imagery (*enactive* and *iconic* representation) prepares children for reasoning meaningfully about 'any triangle' with *symbolic* representation (words and symbols).



In this example, as children physically construct and transform many varieties of triangles with dynamic materials, what they see before their eyes is how lengths of sides may vary endlessly as angles change, how angles may vary endlessly as lengths change and yet, despite all the variety they see, the shapes they are creating all turn out to be triangles.

Dynamic mental imagery deriving from these physical experiences then allows children to speak and to think of 'any triangle' as they imagine an infinitely flexible closed 2D shape with three straight sides.

By stressing the straightness of the sides, that there are always exactly three sides and that the sides fit together to 'close' the shapes they make, and by ignoring the constantly changing lengths of sides and sizes of angles, children become able to imagine 'any triangle' and to reason about it with words and symbols.

Then, having generalized to a stage when their mental imagery allows them to imagine and speak meaningfully about 'any triangle', children are in a position to further generalize that, e.g. 'the angles of any triangle add up to 180°', through further action, imagery and reasoning with symbols.

In the same way that generalizing about numbers is approached, exploring relationships with action and imagery in geometry prepares children for reasoning meaningfully about abstract mathematical objects with symbols.

We reach all the generalizations of mathematics, and the symbols we use to express them, through being active and illustrating our talking about the relationships we are exploring.

Doing mathematics in the world – solving problems

Of course, *doing* mathematics in the everyday world is not simply about making generalizations and using symbols. Crucially, it also involves making *use* of generalizations to solve problems in particular situations.

The generalization '4 × 25 = 100' allows us to predict that the perimeter of a square of side 25 cm will be 100 cm, that the area of a field measuring 4 m by 25 m will be 100 m^2 and that if you save £25 a week for four weeks you will have £100. It can also be very useful to help calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

Children need to be able to readily connect the generalizing that they do with mathematical symbols with everyday world situations in which those generalizations are useful.

In their more general problem solving, children learning, e.g. *how* to divide one number by another, need to learn *when* that operation is useful.

The Numicon teaching materials organize activities into groups based around mathematical themes, such as 'Transformations in the four quadrants' or 'Areas of 2D shapes'.

In the activity group 'Transformations in the four quadrants' (Geometry 3), children begin working on a 4-quadrant coordinate grid. They are introduced to the conventional use of positive and negative numbers to label coordinates and then begin exploring reflection in the x- and y-axes, and translations crossing the axes.

In the activity group 'Areas of 2D shapes' (Measurement 2), children use tangrams to explore the conservation of area when a shape is dissected and its parts rearranged. They then begin finding the area of triangles and parallelograms and derive the formulae for the areas of these shapes. Finally, they move on to finding the area of composite shapes in the context of designing gardens.

In these ways, children's mathematics is introduced, wherever possible, within contexts in which that mathematics is useful. Such contexts help children to 'see the point' of the mathematics they are learning, and prevent mathematics becoming a series of answers to problems they have never had.

Flexibility, fluency, and persistence

Being able to do mathematics effectively also includes being able to remember basic things such as those generalizations known as 'times tables' and number 'facts'. There are many practice activities built into the Numicon written materials that encourage children to develop a regular familiarity with basic facts.

Even more important to effective functioning is flexibility in mathematical thinking and communicating. Three kinds of flexibility are especially useful.

Through being active, children are able to 'invert' their actions; put more simply, this is about 'doing and undoing'.

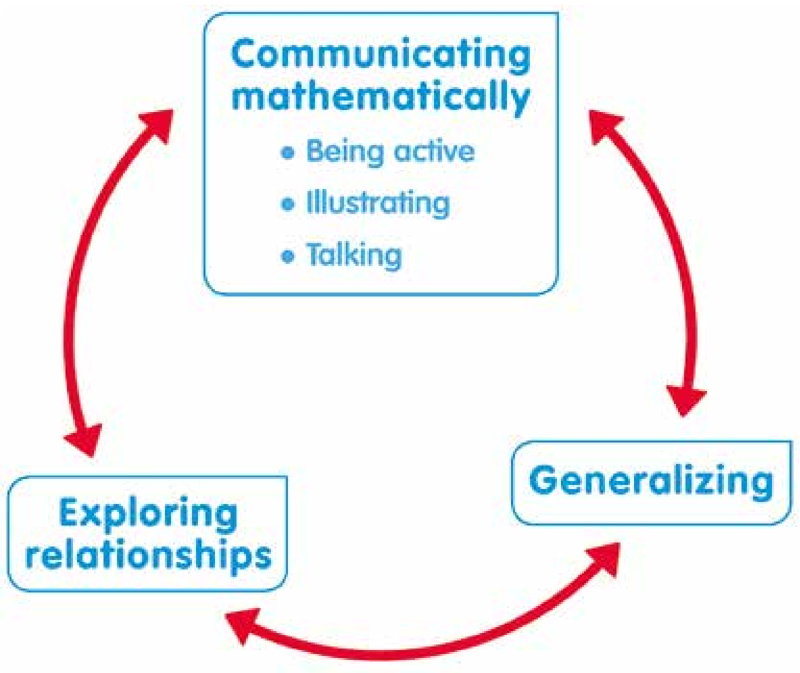
As children have found 'how many' objects there are in a collection by physically grouping them into tens (and hundreds), they are later readily able to 'partition' numbers ('undo' their groupings) when calculating with symbols. As children can combine and separate Numicon Shapes and number rods physically ('do' and 'undo' their actions), they are able to connect adding and subtracting as inverse operations and to check a subtracting calculation by adding.

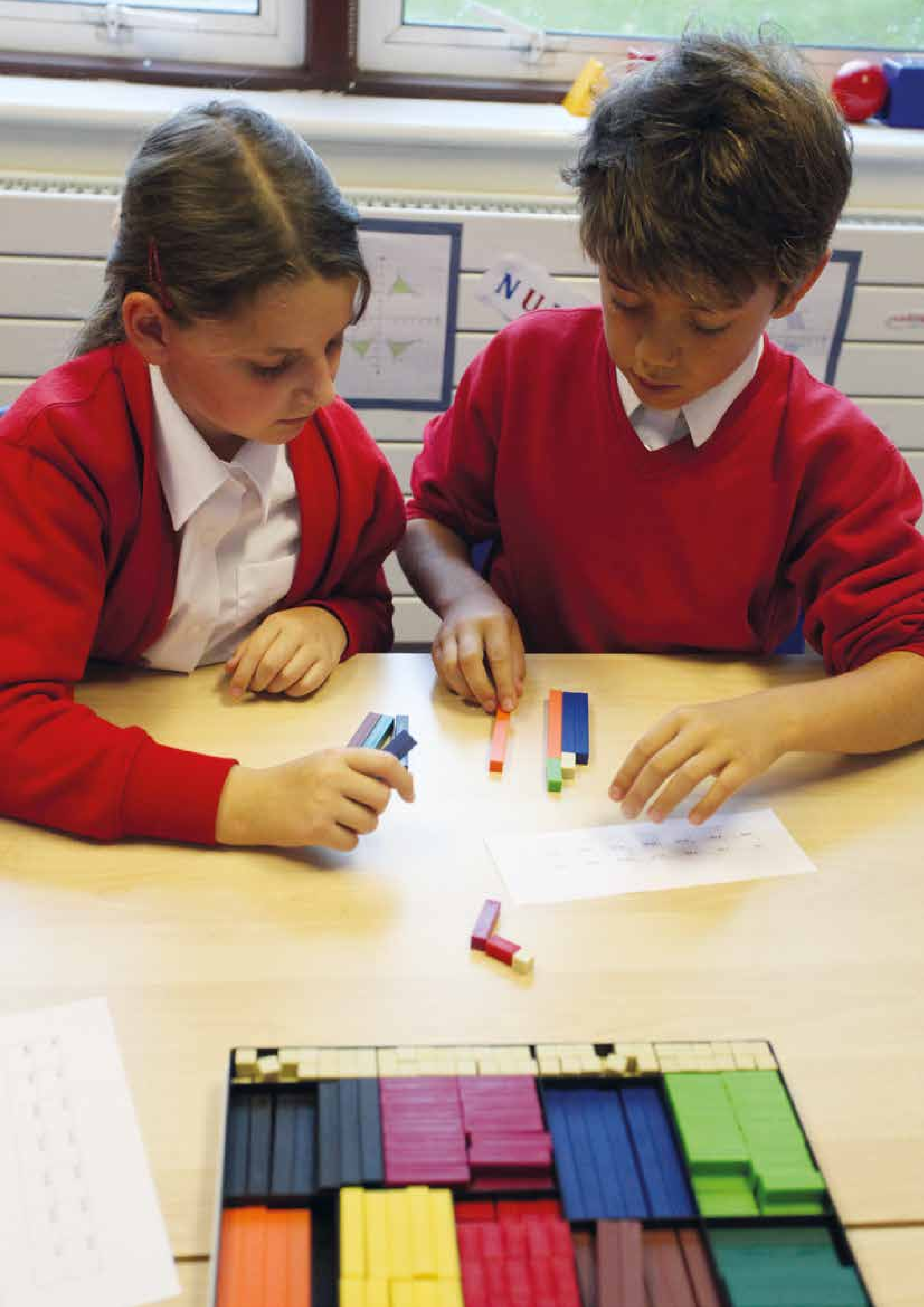
The introducing and encouraging of a variety of ways of calculating means that children are able to choose methods of calculating that suit the particular numbers involved, rather than adopting a standard method for any calculation. Who would want to subtract 1998 from 4673 using a column method, just because the numbers are big?

As Numicon approaches children's use of mathematical symbols with prior actions and imagery, if children become 'stuck' or hesitant while working with symbols, it is always possible for them to recall and return to the supporting activity and illustration from which their original generalizing arose. For example, if they are working on prime numbers, using symbols, they can immediately return to actively illustrating how factors 'go into' their various multiples physically with Numicon Shapes and with number rods. They can follow a similar pattern with any other kind of number relationship. This flexibility is one of movement backwards and forwards between Bruner's *enactive*, *iconic* and *symbolic* forms of representation in their thinking and communicating.

Finally, because Numicon continuously emphasizes communicating mathematically, it is possible to stress to children that all is not lost when they feel 'stuck'. The thing to do, always, when they 'don't know what to do' in a situation is to communicate.

Being active, illustrating and talking about the relationships we are exploring *is* doing mathematics. Persistence – an invaluable quality when doing mathematics – comes from continuing to communicate, with yourself and/or with others, whenever (for the moment) you cannot see where you are going.





Preparing to teach with Numicon Geometry, Measurement and Statistics 6

This section is designed to support you with practical suggestions in response to questions about how to get started with Numicon in your daily mathematics teaching. It also contains useful suggestions on how to plan using the long- and medium-term planning charts as well as information on how to assess children's progress using the Numicon materials.

When beginning to work on Numicon Geometry, Measurement and Statistics 6, as well as reading this section it can be helpful to refer to the suggested teaching progression in the programme of Numicon activities. This, along with the long- and medium-term planning charts, can be found in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*.

In this section you will find overviews of:

How might using Numicon affect my mathematics teaching?

There will be a continual emphasis on communicating mathematically. Once involved in communicating mathematically, children become active, in the sense that they become engaged in 'doing' maths for themselves. They begin to reflect on different points of view and to develop the imagery required to communicate mathematically, as well as the ability to illustrate their ideas.

For geometry and measurement, this imagery and illustration has a particularly active, practical foundation. Children develop and refine their thinking about position, shape, space and size through physical exploration – through making, comparing and manipulating objects or their representations. In their work on measurement, for example, they might consider the effect of various transformations on 2D shapes by using geoboards or shapes drawn on coordinate grids. In this sense, an appreciation of the role of the senses, and dexterity and precision of movement are further aspects of their mathematical communicating. As this communicating becomes established as part of the culture of the classroom, children will increasingly join in conversations with you and their classmates. Maths lessons will develop into dialogue with ready use of imagery to illustrate ideas.

A sense of shared endeavour will emerge as children solve problems through communication and persistence. So, when children feel 'stuck', they will know that the thing to do is to talk about it, to try to explain what the difficulty seems to be and to use illustrations and actions to express the problem. Careful questioning can encourage children to really think about 'difficulty' and to arrive at solutions gradually.

As the focus on mathematical communicating grows, you may become more aware of the words and terms you use in your teaching, and there is a particular emphasis on mathematical vocabulary in Geometry, Measurement and Statistics 6. It is important to use words and terms consistently. Try to encourage other adults in your class – and throughout the school – to use mathematical language consistently. Listen for children using the same mathematical words and imagery to explain their ideas, though to start with they may use them only hesitantly.

In their work on coordinates, for instance, children are introduced to the second, third and fourth quadrants for the first time, having previously worked purely with positive coordinates. They must use the terms positive and negative, x and y, horizontal and vertical correctly in order to give precise instructions.

Similarly, in their work on measurement, children are reminded of the prefix 'kilo-', meaning 'a thousand', from their work on grams and kilograms and are encouraged to apply this knowledge to understand the term 'kilometres', and the relationship of these units to metres, and then centimetres and millimetres.

This increased focus on mathematical communicating will make it easier, through watching, questioning and listening to children as they work, to judge whether they are facing a suitable level of challenge. Activities are structured so that at each stage children encounter a new 'problem' to solve – Numicon aims to encourage children to relish this, to persevere in working through any difficulty, to gain a sense of achievement when it is overcome, and to be excited about and ready to progress to the next challenge.

How can I encourage communicating?

Children respond to the communicating around them. As such, the ways in which you communicate mathematically provide a model for children's communicating. Engaging in dialogue with children, actively listening to what they are saying, and responding sensitively with thoughtful questions, will encourage them to listen to one another and respect each other's ideas.

Make sure that appropriate resources for illustrating ideas in the area of study are freely accessible. For example, when studying volume and capacity, measuring devices such as bottles, jars, jugs or measuring cylinders should be made available to children, or, for 3D shapes, a range of interesting packaging which may be opened up to form nets, geo strips and connectors, geoboards and elastic bands, modelling dough, construction sticks or straws, and squared and isometric 'dot' paper, as well as images and everyday examples of shapes.

Observe how children use the available resources, listen to what they are saying, watch what they are doing and respond with questions and focused praise when you notice active listening and thoughtful questioning. What children do and say will help you to understand what they are thinking.

The activities in Geometry, Measurement and Statistics 6 are also designed to build on a foundation of understanding, skills and knowledge in the areas of number and calculating. Within the Numicon teaching programme, this foundation is provided by Number, Pattern and Calculating 6.

Providing a range of examples and contexts for children's work will also encourage them to develop and refine the imagery they use to 'do' mathematics. Opportunities are highlighted throughout the activities; e.g. creating nets could be presented in the context of food packaging, and newspaper reports could illustrate the need to collect data and present it in a range of graphical formats. Similarly, suggestions are given throughout the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* for practice problems with a wide range of contexts, involving time, distance, mass, and volume and capacity.

In Geometry, Measurement and Statistics 6, children are also continually encouraged to estimate, particularly with regard to capacity, time, distance and volume. Support children to communicate these estimates, either orally or in writing before undertaking any measuring activity. Comparing their estimates against subsequent measurements will increase children's appreciation and understanding of the units of measure that they meet, as well as helping their estimates to become more accurate over time.

Finally, the ways in which children are grouped or paired for working together has an impact on their communicating, so this needs careful consideration (see page 31, 'What about grouping children?').

Using daily routines to encourage communicating

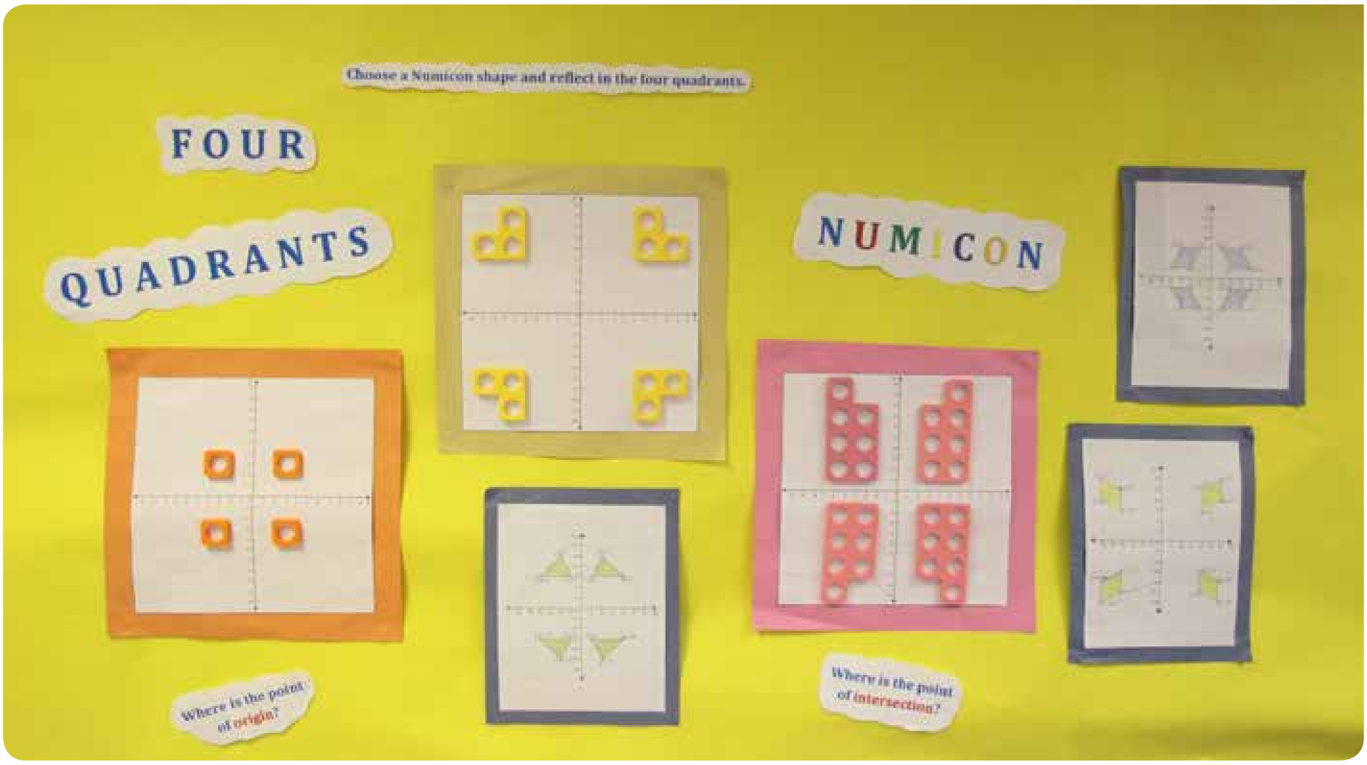
A 'morning maths meeting', perhaps 15 minutes long, has proven to be very successful in encouraging children's mathematical communicating. These meetings are oral and practical and include a small selection of key routines in which children practise rapid recall and develop fluency in their current work. In addition to consolidating knowledge of number and calculation, measurement, geometry and statistics, they can be used as opportunities to provide variety and context. You might discuss with children observations about a mathematically rich object, or solve a mathematical problem that has come up in the class, in the school, in the news, at home or in a story.

You can refer to the whole-class practice activities from the activity groups in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* for ideas. You could also select focus activities from the activity groups to use for class investigating or problem solving.

Children's mathematical thinking and reasoning can also be encouraged at other times during the day. Along with a morning maths meeting, this will help to ensure that they experience the full breadth of the maths curriculum, and that they do not see mathematics as something that only happens in their mathematics lessons.

The daily or weekly timetable and the school year provide meaningful contexts for children to make predictions and to refine their language for and understanding of temporal relationships and units of time. Timetables and calendars will enable children to further engage with this discussion. Calendars, clocks, timers and stopwatches can be used to develop communicating about order, duration and passage of units of time – you could invite children to set an alarm clock or timer to ring at the end of a particular lesson or activity or to use the clock on the classroom wall and a timetable to work out what their next activity or task is, or what time they need to finish a certain task if they are going to complete another task of a set duration by a certain time.

Mathematical thinking can be encouraged in a broad range of classroom tasks and activities. For example, if children are lining up you could ask them to do so in order of height or birth date, and when they are moving between places in the room or school you could ask them to estimate the distance, in a range of units and select the most appropriate, or describe the angles they turn through along the way. In the Geometry, Measurement and Statistics 6 activities, children continue to calculate angles without measuring them by considering their relationships to full turns, triangles, quadrilaterals, parallel and perpendicular lines, and so on. It is therefore valuable to take the opportunity to ask children to consider angles in their environment frequently. They should also be encouraged to distinguish between an angle that exists in a fixed way, such as the corner of a desk, and an angle that is created by them as an amount of turn.



What might the use of Numicon look like in my classroom?

Nearly all of us are acutely visually aware, and children are no exception. A mathematics-rich environment provides valuable learning opportunities; throughout the day, and particularly during mathematics activities, you will find children referring to the imagery and displays around them. There will be number lines and charts on display where they can easily be seen, including a Numicon 1 000 000 Display Frieze, a Numicon 0–1·01 Decimal Number Line, a Numicon $^{-}12$–12 Number Line, a Numicon 0–100 cm Number Line and a place value chart. In a corridor in the school, there may also be a Numicon 0–1001 Number Line to engage with. You can also create displays to reflect children's current mathematical focus, such as a display on transformations or solid 3D volumes. As well as providing a space in which to celebrate children's mathematical work, this might include pictures, books, interesting objects (different types of measuring instruments or scales showing both metric and imperial units when children are learning about converting between the two) and relevant games and challenges (e.g. instructions involving coordinates and transformations for children to follow, perhaps against the clock, to make designs with Numicon Shapes or Counters on a labelled Numicon Baseboard Laminate, a geoboard or a coordinate grid).

Alongside creating a visually rich environment in which to think and communicate about mathematics, the organization of space and resources can encourage children's involvement in doing mathematics. For example, it is useful to set up a mathematics table with an interactive display where children can freely explore Numicon Shapes, number rods, 2D and 3D shapes, measuring instruments, geo strips and other familiar resources. When children are studying geometry and measurement, it is often also helpful to set up some space so that they can try out the equipment and practise the activities they meet in their mathematics lessons. For example, while working on angles children might be given access to protractors, angle software, programmable robots and interesting shapes and angles to explore. In relation to scale, a themed area might contain a variety of objects and resources that children can observe and explore, such as maps and scale drawings, appropriate internet sites with scale images and information about their real dimensions.

Organizing classroom resources systematically, with storage containers numbered and stored in a logical order, ensures that mathematics equipment will be available for children to find and use themselves. They should be familiar with the resources and encouraged to use them freely. The 'morning maths meeting' might be used, early in the year, to explore equipment and how it can be used.

Finally, geometry and measurement activities typically lend themselves to collaborative working, and some will require a suitable space to be set up in advance. As part of your preparations for a new activity group, you may wish to consider how working spaces or desks should be arranged to facilitate this. In particular, some of the activities on exploring circles require a large space such as a hall or a playground. If using the latter, it will be useful to have playground chalks to help children to record their thinking and measurements. (See also the section 'What about organizing resources and space?' on page 30.)

What could using Numicon feel like for children in my class?

In the activities in Geometry, Measurement and Statistics 6, children are asked to think about and create packaging for products, as well as thinking about appropriate gift-wrap to learn about nets, thus seeing the practical importance of the mathematics they are learning. In addition they are asked to think about how mathematics could be used in many real-world contexts such as flooding, factory production, estimating motorway speeds and times and so on. In this way, children are encouraged to recognize that mathematical activity is an ordinary, valuable and interesting part of everyday life and of their ongoing learning about the world. As you engage them in solving mathematical problems, they also learn when this mathematics can be useful.

Within the mathematics-rich environment of the classroom, you are likely to notice children glancing at displays and images to check an idea they are explaining. At other times, you will notice them simply looking thoughtfully at displays – they are likely to be noticing relationships and making connections as they assimilate new ideas.

The open-ended nature of the Numicon resources and activities invites children to experiment and explore, self-correcting as they seek solutions. Children should recognize self-correcting as part of a normal learning process; encouraging them to pursue opportunities to investigate, think, communicate and self-correct will support their confidence.

Working in pairs and within groups also supports children's confidence by encouraging them to share ideas and work things out together before discussing their work with the whole class. Some children are more confident when working with a partner, setting challenges and exploring ideas more deeply than they do when working alone. Children take their lead for listening to each other and sharing ideas from the ways in which the adults in the classroom converse with them and each other. They may need help with taking turns, showing respect for each other's ideas, listening to one another without interrupting, phrasing questions and expressing ideas. Over time, you will find that children become confident in sharing their thinking more and more widely.

With Numicon, children will also know that there is nothing wrong with challenge: it is normal to get 'stuck' – what is important is being able to communicate mathematically about any challenge they face. Children can gradually come to relish challenge when they feel able to persist in the face of it, and gain a sense of achievement when a challenge is overcome. Part of this confidence comes from the ways in which their understanding is built cumulatively by following the suggested progression of teaching activities in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*.

Children also feel increasingly confident when tackling new ideas because they can use a variety of resources and imagery to illustrate problems. Their confidence will be further encouraged as you discuss their ideas with them, helping them to become more aware of what they know and are learning. Such discussion supports children's monitoring of their own learning.

What is the role of imagery in geometry and measurement?

In their work on number, children are asked from a very early point to mentally 'handle' abstract objects, in the form of pure numbers. Numicon's focus on the use of structured materials helps them to approach this difficult task through actions and imagery, and in this way to communicate increasingly effectively about the abstract objects we symbolize with numerals such as '3' or '10' (for more discussion, see 'What is Numicon?' on page 12).

In contrast, children's work on geometry and measurement is rooted in physical objects and how they occupy physical space – their shape, length, mass and so on. They are able to handle these objects directly and immediately: to look at and use the space around them; to draw or make, change, dismantle and remake shapes for themselves; to compare and judge size and quantity in concrete terms – according to how much of something they can fit in their own hand, for example.

Just as for number, though, learning about geometry and measurement requires children to identify relationships and learn to use symbolic representations (words and symbols) that enable them to describe patterns (generalize) – that is, to think and communicate mathematically. To identify any triangle successfully, for example, children must learn to link the word 'triangle' with a generalized set of requirements which can apply in an infinite number of cases; that is, they must be able to distinguish any particular instance of a closed 2D shape with three straight sides. They can then refine their symbolic representation to classify triangles into types: scalene, isosceles, equilateral, right-angled, and so on. For measurement, the reasoning is perhaps more straightforward, but the principles are the same. Identifying which unit of measurement to use requires a generalized understanding of dimensions of length (for instance) and relative sizes. For example, metres would be an appropriate unit for measuring the length of an Olympic swimming pool, whereas millimetres would not.

Children should be encouraged to experience and explore physical space and objects, investigating, constructing and transforming with a variety of resources and materials in a variety of contexts – but in doing this it is the dynamic mental imagery that they build up, rather than the 'raw' number of examples they encounter, that will enable them to generalize and think mathematically. It is important to support the development of children's mental imagery by encouraging a continual emphasis on visualizing, describing and predicting results in their work.

What is the role of investigating in Geometry, Measurement and Statistics 6?

There are three Investigating activity groups for the final term of Geometry, Measurement and Statistics 6, offering opportunities for children to explore new aspects of the mathematics they have been learning in the preceding months and years.

During this investigating it helps if children are encouraged to evaluate their own thoughts and questions, and to share their thinking as it develops, through discussion and debate with others. Children may investigate these new areas either

individually, or in small groups, but should always have the opportunity to discuss and share their current progress, puzzles and findings with others.

While exploring these new areas, children are still being invited to think mathematically. This could involve being systematic (exploring possibilities systematically, in an organized way), generalizing (looking for patterns, working out a general rule and predicting), and being logical (reasoning, to predict, to confirm and to explain).

Very importantly, while investigating these new areas children should have maximum opportunity to convince others about the validity of any new conclusions they reach, and also to question the observations of others. This is essentially how mathematical reasoning develops – in dialogue with others.

Time limits

There is no knowing in advance how long children may take to engage with these new ideas, particularly if it is decided to embrace the possibilities of new software, e.g. spreadsheets. In practice it is useful to set a maximum time limit, say two or three lessons per activity, by the end of which children should be reasonably capable of explaining their work to others. Children who decide they have finished earlier than this could prepare a presentation and then begin to work on another area.

Clearly, when working on this new material, some children will appear to 'achieve more' than others. In order to help *all* children explore these new ideas, it is important to celebrate a wide range of qualities and progress. Being persistent, being systematic and being thorough should be celebrated as much as being imaginative, going further and being articulate.

How much time should I plan to spend on mathematics teaching?

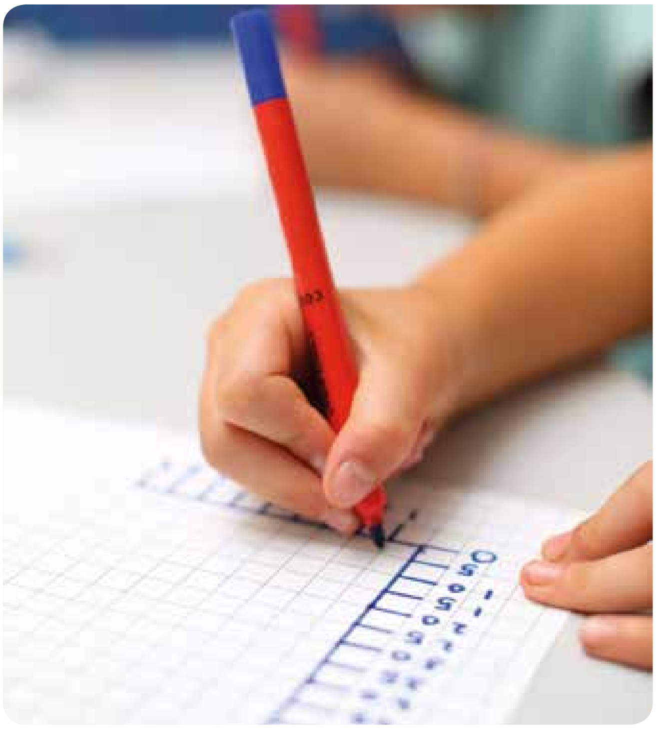
The time spent teaching mathematics during the school day can vary. In addition to a daily mathematics lesson lasting up to an hour and a 'morning maths meeting' lasting about 15 minutes, there are many everyday opportunities to develop communicating about measuring, geometrical ideas and statistics. There will also be many opportunities for estimating or making measurements, and calculating. Taking advantage of these unplanned opportunities helps children to realize that doing mathematics is normal and useful in all sorts of situations, and encourages them to recognize when and how to use the mathematics they have learned.

For planning your teaching time, see the section 'How do I plan in the long- and medium-term using the Geometry, Measurement and Statistics 6 teaching programme?' on page 33.

What format might Numicon mathematics lessons take?

During the lesson introduction, you may use physical materials and other imagery in communicating and discussing ideas with children. You may also use the *Numicon Software for the Interactive Whiteboard*. Children may join in this part of the lesson in lots of different ways: participating in a class conversation, talking with a partner or within a small group, jotting on an individual whiteboard, or using physical resources to explore and show ideas.

When children are working in groups or independently, they may be exploring ideas within the same activity group through a focus activity, an independent practice activity



or investigation, or further work on ideas introduced at the start of the lesson. Different children might use different apparatus in their group or independent work. It is quite likely that a child or children will put forward an idea that is worth everybody considering. You might choose to invite all children to take a moment to reflect on the idea, or make a note to discuss this in the final part of the lesson.

In the final part of the lesson, it is particularly important to encourage all children to reflect on their learning by asking questions of those working at different stages, depending on what you have noticed them doing and saying during the lesson. You may decide to ask the different groups to explain to the rest of the class what they have been doing and what they have noticed. You may have particular points you want to draw to children's attention. You could also suggest what might happen in the next lesson and anything children could think about before then. To help children to reflect, you could ask them to think quietly for a few moments about what they have been doing, and guide their reflection with questions such as:

- Is there anything new you have learned?
- Is there anything you feel particularly pleased about?
- Is there anything you particularly enjoyed?
- Is there something that you found difficult?
- Is there something that you are still puzzling about?
- Is there something you would like to do again?
- Is there something you would like to spend longer on?

What writing or drawing might children do in mathematics if the activities are mainly practical?

Writing and drawing are aspects of communicating mathematically, and children's chosen forms of this communicating may be varied and idiosyncratic at times. Key challenges might involve: choosing sensible scales for graphs, collecting data and choosing the most suitable graphical representation of that data in order to answer questions or make decisions, using dot paper to create 3D drawings or rectilinear shapes, or using protractors, rulers and pencils to create accurate regular and irregular 2D shapes and their nets.

Giving children the choice of how to communicate their ideas can provide useful insights into how they are approaching problems, whether they are working systematically and how they are using conventional notation. It also helps children to formulate, clarify and develop their thinking, and to think of writing and drawing as part of *doing* mathematics.

Accordingly, opportunities for children to communicate on paper are highlighted in the activities wherever this serves a useful purpose. It is a good idea to allow children free access to both square and isometric dot paper (e.g. photocopy masters 7–8), so they can practise drawing 2D and 3D shapes. Some other activity sheets are provided as photocopy masters, but in general it is recommended that children write and draw in their mathematics exercise books. This creates a useful bank of evidence to review how children's mathematical communicating develops over time.

Children's Explorer Progress Books (see page 43 for more information) will also provide an extremely useful source of evidence for monitoring how children are progressing throughout the year.

What about organizing resources and space?

The way in which resources are organized, used and presented and how the available space is set up sends children a strong message about how they are expected to work. (See also the 'What might the use of Numicon look like in my classroom?' section on page 26.)

When preparing the classroom for a mathematics lesson, consider how children will be grouped and how working spaces and desks could be arranged to support particular activities and learning requirements. Set out the equipment where children are going to be working. Refer to the 'Have ready' sections of the activities you plan to teach. The required photocopy masters can be photocopied from the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* or printed from the Numicon Teaching and Assessment Resources on Oxford Owl.

A number of activities in Geometry, Measurement and Statistics 6 require some open space, in the classroom, hall or outdoors, e.g. for thinking about the perimeter of circles in Geometry 2. Again, refer to the 'Have ready' sections of the activities in order to arrange for suitable space in advance. A large water butt outside would be an ideal water source to support children's practical work on volume and capacity, and some playground chalks would also be beneficial where appropriate.

It is also useful to consider collecting real-life examples well in advance – including images and objects – which will be needed to provide context for children's work. For Geometry, Measurement and Statistics 6, these might include: small food boxes, mirrors, gift boxes, images of running tracks, pictures of a range of simple flag designs (or actual flags), examples of reflected, translated or rotated patterns (such as fabric, gift wrap or wallpaper) and thermometers.

Children may help themselves to any further equipment they need from the class mathematics resources. As children become used to working with Numicon and other mathematical apparatus, they can begin to collect the equipment they need for an activity by following a list (annotated, if necessary, with pictures, symbols or container numbers) and thinking for themselves about any further items they may want to use.

Other ideas for organizing physical resources are:

- providing smaller containers of equipment for children working individually or in pairs
- using sorting trays with separate compartments to allow children to access resources such as numeral cards or measuring equipment, to save space and keep the resources organized
- storing paper resources which children may want to use, such as number lines and geo strips, by hanging them from hooks (punch a hole in the end, if necessary).

What about grouping children?

By the time children are working on activities from the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, some children will be comfortably fluent with the mathematics they have learned in previous years; they will be able to engage confidently with the new ideas they are meeting and communicate their ideas clearly. Others will need more time to consolidate their understanding of ideas met earlier, plenty of practice to develop fluent recall of key number facts and more exposure to the new mathematical words and terms they are meeting, before they are able to communicate their ideas confidently. These differences need to be taken into account when grouping children and planning lessons.

Some schools respond to this by putting children into sets of similar ability. It is important to be aware that there are risks involved in ability setting that can have negative impacts, such as: of putting an artificial ceiling on expectations of children's ability and achievement; of creating an air of complacency in higher ability sets; of children in lower ability sets seeing themselves as failing in maths. Children placed in either a high or low ability group, irrespective of their ability, are likely to take on characteristics of that group: misplacement can therefore result in children underachieving.

Mixed ability classes provide opportunities to be flexible about grouping so children can sometimes work in mixed-ability groups and at other times with children working at a similar level. Whatever the policy is in your school, bear in mind that it is important to vary groups from time to time to ensure that children do not always work with the same partner.

How do I prepare for teaching mathematics lessons using Numicon?

Understanding the mathematics yourself

No one carries all the details involved in doing, e.g. metric and imperial conversions, or transformations, in their heads all the time; we all need to remind ourselves what particular areas of mathematics are about before we introduce them again to children. Before teaching an activity group, read the relevant 'Key mathematical ideas' sections to prepare for your teaching. If you want to do more research on an area of mathematics, you may wish to consult other sources, for example Derek Haylock's *Mathematics Explained for Primary Teachers* (5th ed., 2014).

Next, consider what generalizing there is in the activity group. Ask yourself: 'What patterns in the work children are doing will they have to notice in order to progress?' For example, 'Is this where children notice that the prefix "milli" signifies "one thousandth", whichever unit of measure it is applied to?' or 'Is this where children might recognize that the interior angles of *any* triangle add up to 180°?'

When children notice things, be prepared to keep asking: 'Will that always work?', 'What if those quantities were different?', 'Would that work with a different shape?', 'Will that ever work?', 'When does that work?', and 'What never works?'

Appreciating the contexts

The 'Educational context' on the introductory page for each activity group will help you to see how the ideas involved fit into the continuum of children's learning about Geometry, Measurement and Statistics.

After reading this educational context, consider when this learning may be useful. Children don't just need to know *how* to do this mathematics; they need to know *when* to do it as well. How can you help them to spot when this general mathematics applies to a particular situation?

Think about the kinds of context offered in the activity group, and ask yourself whether the mathematics is useful in particular kinds of real-world situations, or whether it helps with doing other mathematics. It can be helpful to think up one or two contexts of your own, so that it is clear what the point of doing this mathematics is.

It is important to keep in mind that geometry, measurement and statistics themselves are key contexts within which numbers are practically useful – we use numbers to help us specify quantities, positions, directions, and magnitudes. Without being able to specify these aspects of our physical universe precisely it is difficult to imagine how human life could progress at all.

Selecting and adapting activities

You know the children in your class and the materials and spaces which are available to you. You will be best placed to select which activities are most appropriate and adapt them creatively to suit the needs of your class.

Read all of the activities in an activity group and identify what each activity contributes overall, as well as the resources and preparation involved. Then try the activities. Some might be revision for your children; others might involve ideas, illustrations or techniques that are completely new. Some activities will be invaluable practice. You might think an activity will be too easy or too difficult, so think about how

you might adjust the level of challenge. Be flexible and adapt what is available for your children in the light of what they can already do.

How do I deal with children who are stuck?

It is important for children to know that there is nothing wrong with challenge; it is quite normal to get 'stuck' when working on a non-routine problem. The author John Holt wrote that 'The true test of intelligence is not *how much* we know how to do, but *how we behave* when we don't know what to do.'[1]

What is important is that children are encouraged to communicate mathematically in the face of challenge, and you can help them to do this by asking questions such as: 'How did you start?', 'Were there any parts you did know how to tackle?', 'Where did it become difficult?' and 'What do you know that you haven't used yet?' This can help them to explain what their difficulty seems to be, and to use illustrations and actions to express their problems. In time, children will start to feel that they can ask themselves these sorts of questions so that they become more willing to persist through difficulty.

Eventually the class culture will become one in which children are willing to deal with challenge because they can remember a sense of achievement when they do so. Encouraging children to express and deal with challenges will also help them to respond positively if they are stuck in a test or exam, when they will need to explain the problem to themselves silently and visualize mental imagery. Aim for an environment where children accept the fact that everything is difficult until they can understand it themselves, after which it becomes easy *for them*.

How do I plan in the long- and medium-term using the Geometry, Measurement and Statistics 6 teaching programme?

The plan–teach–review cycle applies to Numicon, just as it applies to all effective mathematics teaching. Four important features of Numicon support this cycle.

First, the Numicon teaching programme – the recommended order of the activity groups – is structured progressively. A chart showing the programme appears in the long- and medium-term planning section of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* (also available as an editable version on the *Numicon Planning and Assessment Support* on www.oxfordowl.co.uk).

Second, practice and discussion activities are included in each activity group, for individual, paired and group work.

Third, accurate assessment is enabled through children's practical work with physical resources and imagery, and their mathematical communicating in conversation and on paper. This assessment will, in turn, help with planning.

Finally, 'using and applying' does not need to be planned separately. This is because all activity groups are based around problems to be solved, and because the cumulative nature of the programme means that children are using their earlier learning every time they face new ideas.

1 Holt, J. (1982) *How Children Fail* (revised edition). New York: Perseus Books

The teaching programme for Geometry, Measurement and Statistics 6 is arranged into two strands, 'Geometry' and 'Measurement', with a series of activity groups within each. Both include coverage of statistics, in the form of data recording, presentation and interpretation. There are also three activity groups designed to provide extra problem solving practice, though it is expected that problem solving is not seen in isolation, but rather as a key reason for learning to think mathematically, and integral to all the activity groups.

The long-term plan on page 17 of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* shows the recommended order of the activity groups. It has been carefully designed to scaffold children's understanding, so that they are able to meet the challenges of each new idea. For instance, children would not be expected to learn about the 24-hour digital clock without being able to tell the time to the nearest minute using the 12-hour digital clock.

It should be noted that this structure has been designed together with the progression of the Numicon Number, Pattern and Calculating 6 teaching programme. The long-term plan included on the Numicon Planning and Assessment Support contains suggestions for integrating the Geometry, Measurement and Statistics 6 activity groups with those from the *Number, Pattern and Calculating 6 Teaching Resource Handbook*.

The medium-term plan on pages 18–21 of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* gives expected coverage over the course of the year and also lists the activities and learning opportunities for each group.

You may decide to follow the long- and medium-term plans as they stand. You may also find that you need to split some of the larger activity groups and return to them later.

There are summary charts showing focus activity titles and learning opportunities for each activity group in the long- and medium-term planning section of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* and in the Numicon Planning and Assessment Support. You may find these useful for incorporating Numicon activities into your existing mathematics plans, should you decide not to follow the Numicon long-term plan for teaching the activity groups.

The parts and structure of each activity group are highlighted in the key to the activity groups on pages 38–39 of this Implementation Guide (this is also included in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*).

Each activity group begins with a relatively 'low threshold' focus activity designed to encourage and support confidence and ensure that all children are included. The remaining focus activities are designed to help children progressively develop their ideas around the mathematical theme of the activity group. The focus activities are designed for whole-class and group teaching. Some may be taught quite quickly to the whole class as an introduction to be explored later with a focus group. Opportunities for reasoning through challenging questions and problems are provided throughout. Indeed, three of the activity groups (Geometry 2 and Measurement 3 and 4) contain series of problem solving activities using a range of measures in new contexts.

Ensure that activities contain an appropriate level of challenge for the children in your class. Include activities that support children in becoming more confident; encourage them to work more effectively through practising, and celebrate what they are able to do. Check that there is scope for children to take the activity further. You can increase

challenge by asking more challenging questions, either with specific individuals or to the class more generally. Your questions should relate to what you have noticed the children doing and saying as they work.

How long should I allow for teaching each activity group?

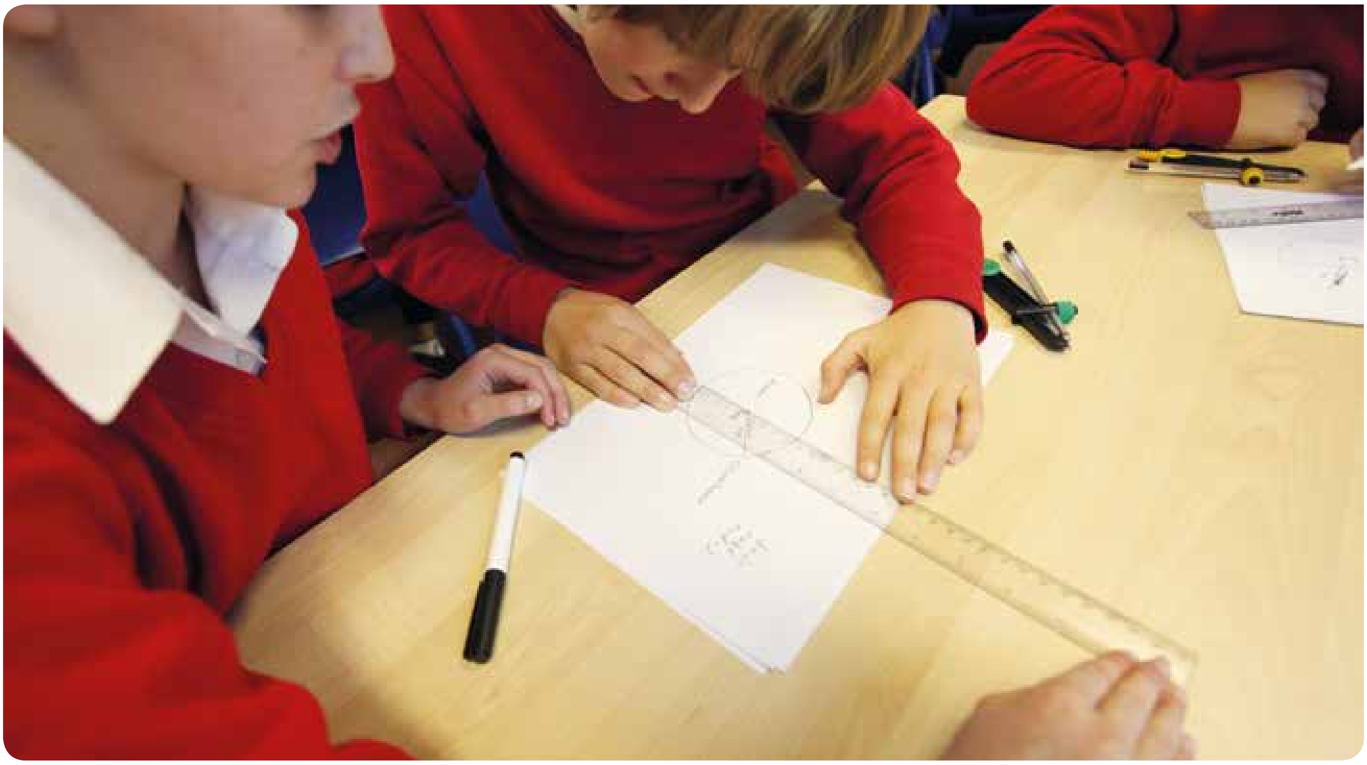
The Numicon teaching activities for Geometry, Measurement and Statistics 6 primarily address the shape, position, measurement and statistics areas of the curriculum, along with some aspects of number.

There are 10 activity groups within the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*. Each of these activity groups can form the basis of approximately one week's work around an area of mathematics, although this will vary depending on both the activity group and the children. The depth and reach of the mathematics that children are meeting in Geometry, Measurement and Statistics 6, and the subsequent extended nature of the activities, mean that, at times, children will need longer to work on them. This time may be outside the usual mathematics lesson or it may take several lessons. You will also gauge from children's responses whether they need a longer timescale for some of the more difficult ideas, so it is important that you are flexible in the amount of time you allow for different activity groups.

Look through the long- and medium-term plan for Geometry, Measurement and Statistics 6 to assess how much time you think will need to be given to each activity group. Careful consideration will need to be given to how much of each activity group is essential for the children in your class to complete before moving on. You might choose to make selections from an activity group and integrate the material with work on other areas of mathematics, for example teaching activities on using formulae to find areas of shapes (*Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, Measurement 2) with work on using letters for variables (from the *Number, Pattern and Calculating 6 Teaching Resource Handbook,* Pattern and Algebra 4). It will also be crucial for you to leave sufficient time for the activity groups at the end of the Numicon teaching programme so that children don't miss out on content that happens later in the school year.

It is unlikely that you would expect all the children to do all the activities in every activity group but the detailed progression and range of focus activities is there to provide flexibility for teachers to exercise their professional judgment as to which children need to work carefully through the earlier activities in a group and those who may move on quickly to the later, more challenging activities. If children are having difficulties getting to grips with an aspect of the mathematics you are teaching, refer to the detailed progression order of the long- and medium-term planning charts in the Teaching Resource Handbook to find earlier coverage of the topic in Geometry, Measurement and Statistics 6, or in earlier years.

As you teach the activities you will find that a few children will move on very quickly and you will be able to combine two or sometimes even three activities within one teaching session. These children may well complete all the activities in the group but careful questioning is advised in order to check that understanding is sufficiently deep.



Other children may need longer to establish secure understanding but will cover several of the activities and benefit from returning to finish the activity group after a week or so. This has the advantage of reminding children about ideas they have met earlier and gives you useful opportunities to review what they have remembered. For more assistance, refer to the medium-term planning guide in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* (pages 18– 21), and consider when might be the best time to ask children to work on the relevant *Explorer Progress Book* pages.

You will see that the progression of activity groups in the medium-term plan is punctuated at intervals by assessment milestones. When considering which activities to utilize from an activity group, refer to the relevant milestone statements from the medium-term planning to guide you. The milestones pick out those aspects of mathematics that are absolutely essential for children's progress, and so specify what they cannot afford to miss.

How should I choose activities?

It is important for children of all abilities that activities provide appropriate levels of challenge.

The educational context and learning opportunities on the introductory page of each activity group give you an overview of the ideas children will meet and the learning to be built upon. If your assessments tell you that your children are not yet ready for the activity group, you can look back through earlier activity groups in the same strand (and even in earlier years) to find appropriate activities. You may need to adapt your chosen activities to suit your children, for example if the number content is too challenging, it may be appropriate to adjust the number range used in the activities; alternatively, consider whether it is best to revisit activities later, once children are more secure in the required knowledge.

Each activity group starts with a relatively 'low threshold' activity designed to be accessible to all children. In a mixed-age class you may need to modify the work for younger children and assess how they respond. You may decide that some children are ready to go straight to more challenging activities, later in the activity group.

The open-ended nature of the activities and the emphasis on mathematical thinking means that there is always room for children to take activities further. You may decide to increase the challenge by planning specific questions that extend the reach of an activity. You might also challenge children to create their own similar problems for others to solve.

If children are having difficulties working with some new ideas you may need to work for longer on the earlier activities in a group (or even activities from an earlier activity group in the *Geometry, Measurement and Statistics 6, 5 or 4 Teaching Resource Handbooks*). Once children have a sound understanding of the mathematical ideas involved, you can return to the later activities in the activity group to extend their reasoning in new or more complex situations.

How can I support children to develop and maintain fluency?

Each activity group includes suggestions for whole-class and independent practice and discussion to help children to develop fluent understanding of the ideas they are meeting. You can select from these to give children appropriately challenging opportunities that will help them consolidate and progress.

In the weeks after you have taught an activity group, continue to encourage children's fluency by drawing on these suggestions for whole-class practice and continue to set problems. This can be done during 'morning maths meetings', at odd times of the day, or in the mathematics lesson.

The Explore More Copymasters provide further opportunities for children to practise and discuss at home the ideas they have been working on at school. The 'morning maths meeting' also provides an excellent opportunity for practice and discussion.

Using the activity groups

The first page of each activity group is clearly coloured according to the **strand** it appears in (Geometry – green, Measurement – purple, GMS Investigating – light purple; statistics is covered within these strands through appropriate contexts). The title and the numbering of the activity group allow you to easily identify the content of the activity group and how far through the strand you are.

The **key mathematical ideas** clearly highlight the important ideas children will be meeting within each activity group.

Activity groups have been extensively trialled in the classroom, so the **learning opportunities** come from real classroom experiences. They are designed to help children to develop their understanding of the key ideas of an activity group.

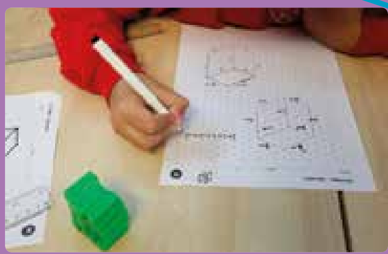
The **educational context** gives a clear outline of the content covered in the activity group, for example how it builds on children's prior learning, how it connects with other activity groups and the foundation it establishes for children's future learning.

The **assessment opportunities** signal key information to 'look and listen for' that indicate how much of the focus activities children have understood.

Key mathematical ideas Length, Area, Volume and capacity, Working in 2D and 3D, Scaling, **Measurement**

Volume and scaling

4



Educational context

This activity group builds on the work on capacity and volume from the *Geometry, Measurement and Statistics 4 Teaching Resource Handbook*, Measurement 5, and extends the work on solid and liquid volume from *Geometry, Measurement and Statistics 5 Teaching Resource Handbook*, Measurement 4.

As with the work on area in *Geometry, Measurement and Statistics 6*, Measurement 2, children are encouraged to discover a formula for calculating volume, which they then put to use in solving practical problems.

In *Geometry, Measurement and Statistics 5 Teaching Resource Handbook*, Measurement 4, Activity 4, children developed an understanding of the equivalence between cubic centimetres and millilitres and converted between millilitres or litres and cubic centimetres. They can use this understanding for the work with cubic millimetres, metres and kilometres in this activity group.

The later activities continue work with scaling, and the effect of scaling on length and area. For support with identifying equivalent ratios, refer to the *Number, Pattern and Calculating 6 Teaching Resource Handbook*, Calculating 7.

The work in this activity group is demanding with plenty of hard thinking for children. Allow plenty of time for discussion and reflection.

Learning opportunities

- To calculate, estimate and compare the volumes of cubes and cuboids using standard units, including cubic centimetres (cm³) and cubic metres (m³), and extending to other units, e.g. mm³ and km³.
- To explore and relate different units of volume.
- To recognize when it is possible to use formulae to calculate volumes of shapes.
- To solve problems involving calculating with, and converting between, units of measure and using decimal notation up to three decimal places.
- To investigate the effect of scaling on the lengths, surface areas and volumes of shapes

Words and terms for use in conversation

length, millimetres, centimetres, metres, kilometres, area, square millimetres/centimetres/metres/kilometres, volume, cubic millimetres/centimetres/metres/kilometres, millilitres, litres, dimensions, one-/two-/three-dimensional, width, height, formula, equation, orientation, square/cube numbers, scale, scale factor, enlarge, reduce, ratio, proportion

Assessment opportunities

Look and listen for children who:

- Use the words and terms for use in conversation effectively.
- Can explain how to calculate the volume of a cuboid and that the three dimensions can be multiplied in any order.
- Know that 1 cm³ is a measurement of solid volume and is equal to 1 ml, the liquid volume equivalent.
- Can compile a list of equivalences for metric units of length, area and volume.

Explorer Progress Book 6, pp. 10–11

After completing work on this activity group, give small focus groups of children their Explorer Progress Books and ask them to work through the challenges on the pages. As children complete the pages, assess what progress they are making with the central ideas from the activity group. Refer to the assessment opportunities for assistance.

Explore More Copymaster 7: Curious Cuboids

After completing work on Activity 2, give children Explore More Copymaster 7: Curious Cuboids to take home.

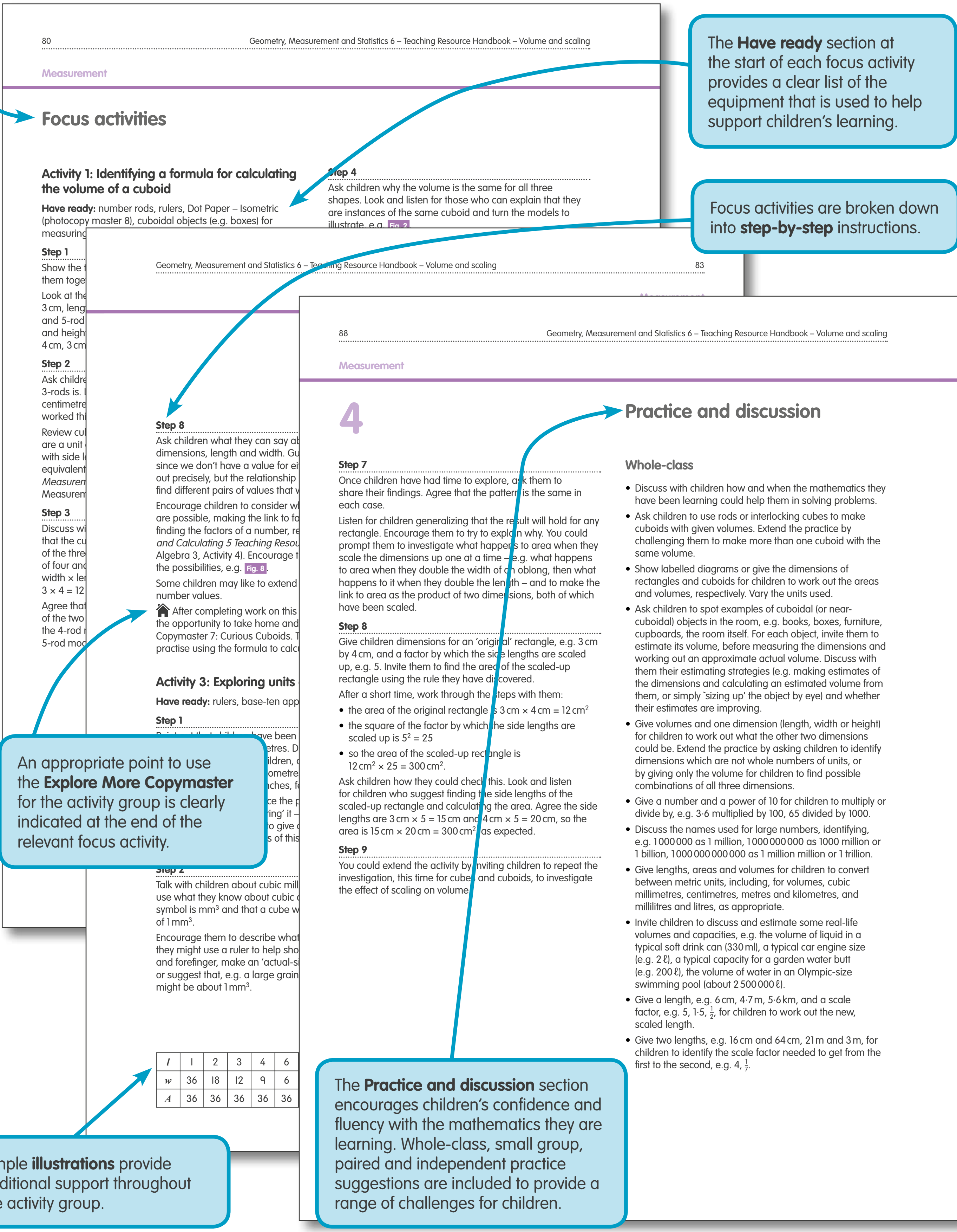
Explore More Copymasters provide an opportunity for children to practise the mathematics from the activity group outside of the classroom through fun, engaging activities.

Clear links are made to the **Explorer Progress Book**. The book provides an invaluable chance to see children's thinking, monitor their progress and assess how much of the activity group they have understood.

Each activity group includes several **focus activities**, each clearly titled to show the specific learning points addressed by the activity. The first focus activity is a relatively 'low threshold' activity allowing all children to engage with the work. Focus activities then build progressively to a 'high ceiling' that provides challenge and allows for some differentiation within the same activity group.

The **Have ready** section at the start of each focus activity provides a clear list of the equipment that is used to help support children's learning.

Focus activities are broken down into **step-by-step** instructions.



88 Geometry, Measurement and Statistics 6 – Teaching Resource Handbook – Volume and scaling

Measurement

4

Step 7

Once children have had time to explore, ask them to share their findings. Agree that the pattern is the same in each case.

Listen for children generalizing that the result will hold for any rectangle. Encourage them to try to explain why. You could prompt them to investigate what happens to area when they scale the dimensions up one at a time – e.g. what happens to area when they double the width of an oblong, then what happens to it when they double the length – and to make the link to area as the product of two dimensions, both of which have been scaled.

Step 8

Give children dimensions for an 'original' rectangle, e.g. 3 cm by 4 cm, and a factor by which the side lengths are scaled up, e.g. 5. Invite them to find the area of the scaled-up rectangle using the rule they have discovered.

After a short time, work through the steps with them:

- the area of the original rectangle is 3 cm × 4 cm = 12 cm²
- the square of the factor by which the side lengths are scaled up is 5² = 25
- so the area of the scaled-up rectangle is 12 cm² × 25 = 300 cm².

Ask children how they could check this. Look and listen for children who suggest finding the side lengths of the scaled-up rectangle and calculating the area. Agree that the side lengths are 3 cm × 5 = 15 cm and 4 cm × 5 = 20 cm, so the area is 15 cm × 20 cm = 300 cm² as expected.

Step 9

You could extend the activity by inviting children to repeat the investigation, this time for cubes and cuboids, to investigate the effect of scaling on volume.

Practice and discussion

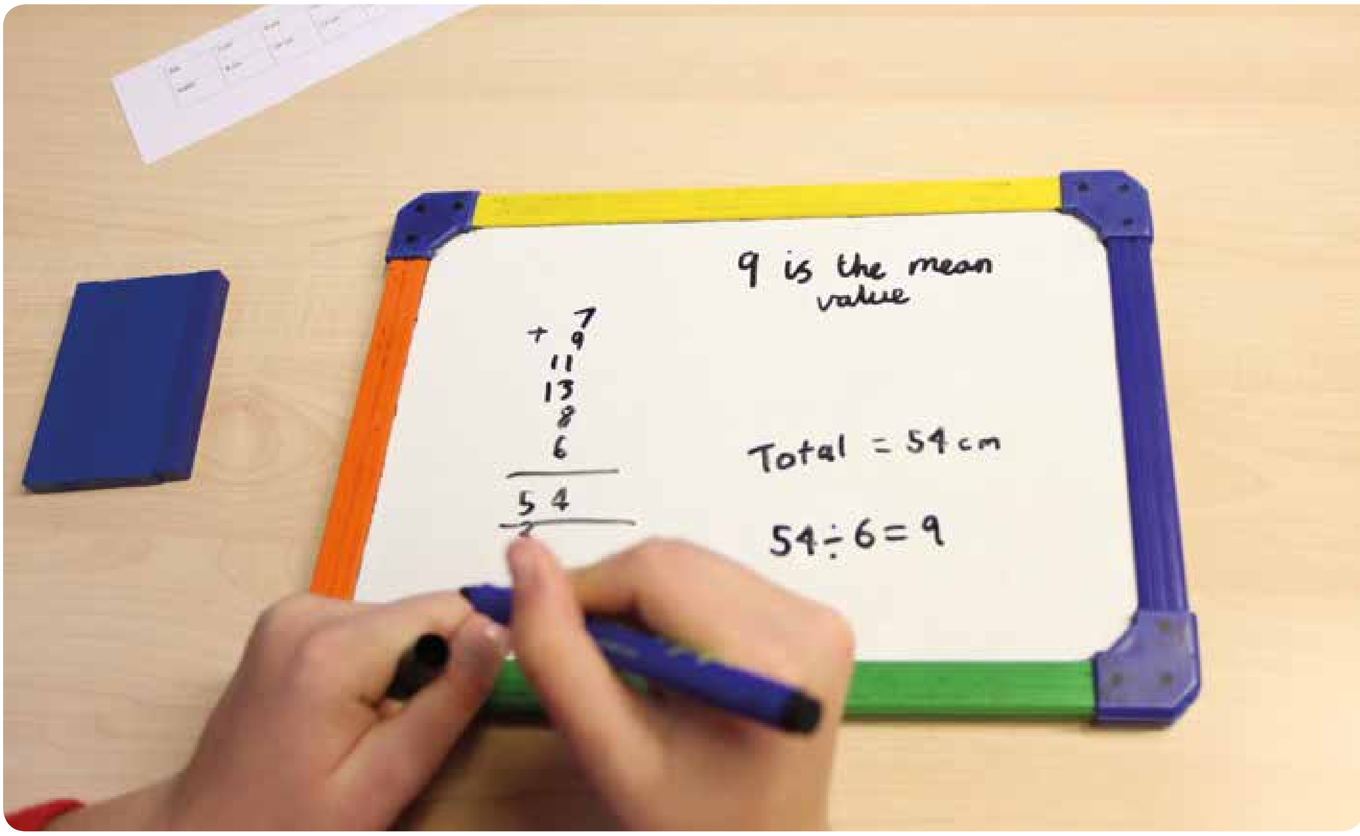
Whole-class

- Discuss with children how and when the mathematics they have been learning could help them in solving problems.
- Ask children to use rods or interlocking cubes to make cuboids with given volumes. Extend the practice by challenging them to make more than one cuboid with the same volume.
- Show labelled diagrams or give the dimensions of rectangles and cuboids for children to work out the areas and volumes, respectively. Vary the units used.
- Ask children to spot examples of cuboidal (or near-cuboidal) objects in the room, e.g. books, boxes, furniture, cupboards, the room itself. For each object, invite them to estimate its volume, before measuring the dimensions and working out an approximate actual volume. Discuss with them their estimating strategies (e.g. making estimates of the dimensions and calculating an estimated volume from them, or simply 'sizing up' the object by eye) and whether their estimates are improving.
- Give volumes and one dimension (length, width or height) for children to work out what the other two dimensions could be. Extend the practice by asking children to identify dimensions which are not whole numbers of units, or by giving only the volume for children to find possible combinations of all three dimensions.
- Give a number and a power of 10 for children to multiply or divide by, e.g. 3·6 multiplied by 100, 65 divided by 1000.
- Discuss the names used for large numbers, identifying, e.g. 1000 000 as 1 million, 1000 000 000 as 1000 million or 1 billion, 1000 000 000 000 as 1 million million or 1 trillion.
- Give lengths, areas and volumes for children to convert between metric units, including, for volumes, cubic millimetres, centimetres, metres and kilometres, and millilitres and litres, as appropriate.
- Invite children to discuss and estimate some real-life volumes and capacities, e.g. the volume of liquid in a typical soft drink can (330 ml), a typical car engine size (e.g. 2 ℓ), a typical capacity for a garden water butt (e.g. 200 ℓ), the volume of water in an Olympic-size swimming pool (about 2 500 000 ℓ).
- Give a length, e.g. 6 cm, 4·7 m, 5·6 km, and a scale factor, e.g. 5, 1·5, $\frac{1}{2}$, for children to work out the new, scaled length.
- Give two lengths, e.g. 16 cm and 64 cm, 21 m and 3 m, for children to identify the scale factor needed to get from the first to the second, e.g. 4, $\frac{1}{7}$.

An appropriate point to use the **Explore More Copymaster** for the activity group is clearly indicated at the end of the relevant focus activity.

Simple **illustrations** provide additional support throughout the activity group.

The **Practice and discussion** section encourages children's confidence and fluency with the mathematics they are learning. Whole-class, small group, paired and independent practice suggestions are included to provide a range of challenges for children.



Planning and assessment cycle

Here is a guide to show how planning can be informed by your assessments of children's understanding.

1. Choose an activity group	Before embarking on the activity group, review the educational context, consider your initial assessments of children and decide whether there is a need to revise any ideas children have met earlier.
2. Choose a focus activity	If this is the first lesson using the activity group, start with an early 'low threshold' activity to include all children. Before the lesson, check the learning and assessment opportunities, brief teaching assistants and prepare resources.
3. Choose the practice activities	**Independent groups:** Refer to your assessment notes and allocate suitable practice activities (found at the end of the activity group). **Focus teaching groups:** Refer to your assessment notes and the learning and assessment opportunities from the activity group and allocate a focus activity.
4. Plenary session (normally during and at the end of lessons)	Think about the important ideas that children meet in the lesson, particularly any generalizations that you want children to make. Plan questions to prompt discussion and encourage children to reflect on ideas they may have learned. Refer to the practice and discussion section of the activity group to find suggestions for whole-class practice.
5. After the lesson	Reflect on how children have responded in the lesson and note any significant steps on your assessment records. Use what you notice to determine the plan for the next lesson. The whole-class practice suggestions will also help children to develop the ideas they have learned in the lesson. At some point after children have completed the activity group, ask them to complete the relevant pages of their Explorer Progress Book. This will allow you to assess how well they have retained the information they have been learning. It will also give you a chance to see how well they are able to apply this knowledge when faced with a 'non-routine' problem.

Creating short-term plans

Here is a template for how you might create a short-term plan. An editable version of this template can be found on the Planning and Assessment Support.

Creating short-term plans

	Warm-up	Main teaching focus	Focused group work with the class teacher or teaching assistant	Independent work	Plenary
Activity number/title	Select activities from the **Whole-class practice** section. This could be from a previous activity group to review and refresh children's previous learning.	Select one of the focus activities from the activity group, matched to the needs of the children. Place the **activity number/title** of the chosen focus activity in your short-term plan.	Decide whether to: • select the next **activity number/ title** from the focus activities in the activity group – place this in your short-term plan; or • consolidate the activity covered in the main teaching focus.	Decide whether to: • choose activities from the **Independent practice** section for groups, pairs or individual children – make notes on your plan or work from the Teaching Resource Handbook; or • select a focus activity for groups to work on independently – place the relevant **activity number/ title** in your short-term plan.	Encourage children to have a reflective conversation to draw together what has been learned in the lesson. Select activities from the **Whole-class practice** section.
Learning opportunities	Place the selected learning opportunity (or opportunities) from the chosen activity group summary in your short-term plan.				
Notes and educational context	Decide whether to: • use the activity directly from your Numicon *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, or • draw on the Teaching Resource Handbook to make your own notes for teaching the activity.	Decide whether to: • use the focus activity from your Numicon *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, or • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	Decide whether to: • use the focus activity from your Numicon *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, or • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity. If working with a teaching assistant, you may want to select the relevant **Educational context** from the chosen activity group.	Decide whether to: • use the practice or focus activity from your Numicon *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, or • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	
Words and terms	Decide which **words and terms** you will use in **conversation**. Place these in your short-term plan.				
Resources	Prepare any resources you may need for the activity. Use the **Have ready** section at the beginning of the focus and practice activities.				
Assessment opportunities	Select from the chosen activity group summary the **assessment opportunities** that you and the teaching assistant will be looking and listening for in the different parts of the lesson. Place these in your short-term plan. Remember to note whether children know when to use their understanding.				

How can I assess children's progress for teaching?

Assessing mathematics using Numicon involves making judgments about developments in children's mathematical communicating – both receptive and expressive. You need to know which are the key developments to look for: check the assessment opportunities given on the introductory page of each activity group and consider how the achievements listed would show up in children's mathematical communicating. Look for developments in children's actions (what they do and notice), the imagery they use and respond to, and their use of, and responses to, words and symbols in their conversation.

It is also important to notice children's fluency. For example, when their communicating is stilted, when it is punctuated by gaps and hesitations, and when it flows consistently and well, suggesting a strong understanding of well-established ideas and the connections between them. Remember that speed is not the same as fluency, though it may often be a by-product of fluency. For example, when we become familiar with performing varied calculations we do tend to work faster, but just trying to work fast often leads to errors.

Assessment should be as open as possible so that children can communicate as much as possible. It is through their mathematical communicating that you will gain a real insight into how children are thinking. This will enable you to make the most accurate assessment of their progress.

Specific challenges for the purposes of assessing are provided in the form of the Explorer Progress Book (see page 43). Children cannot pass or fail these assessment tasks – they simply respond in their own ways. How they approach the tasks informs you about their mathematical communicating and gives you an opportunity to 'see' their thinking through the imagery they use. This insight makes it easier to gather meaningful and accurate evidence of where children are. Preparing for formal test situations is something different, and is addressed on page 44.

Specific indications of children's progress

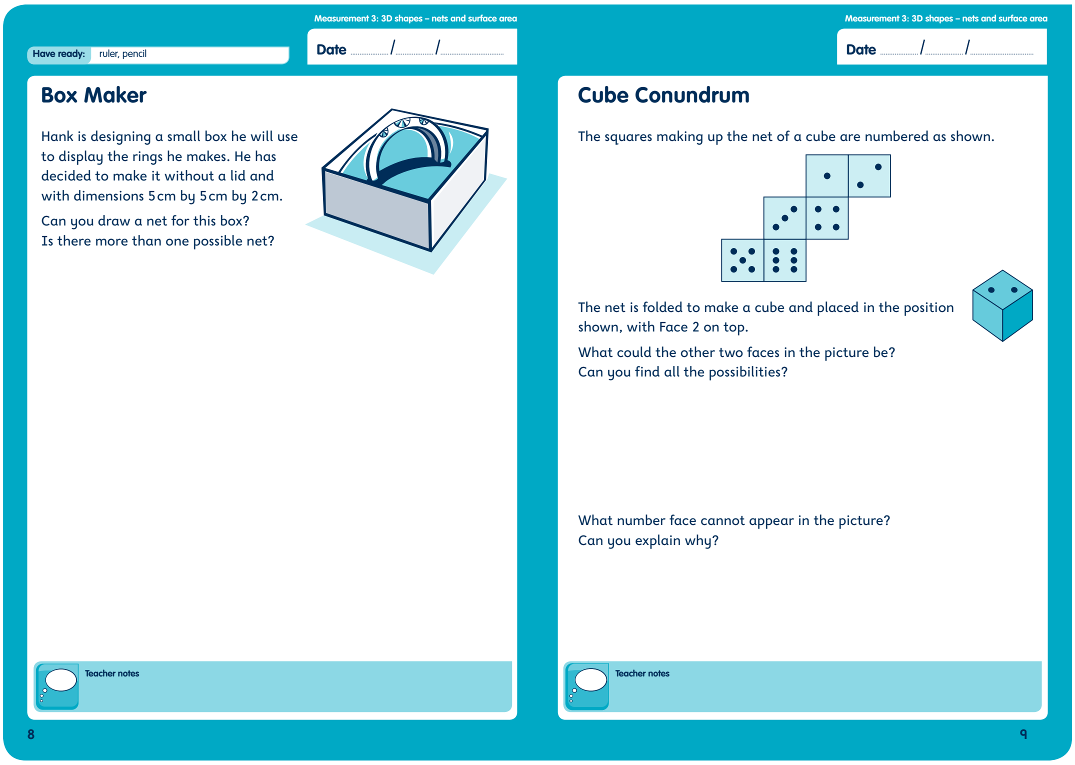
Each activity group lists several assessment opportunities that point to key achievements to look for as children work on the activities. All of these achievements will be observable in children's actions, imagery and conversation as they progress.

Familiarize yourself with the assessment opportunities before you begin teaching an activity group. Use them to help guide your interactions with children, and also as indicators of progress and sources of information to help you to group children and plan your teaching as you move on to further activity groups.

Suggestions for what to 'look and listen for' are given within each activity. Focus on children's communicating and ask yourself whether they know both how to do the mathematics they are learning and when to use it.

You will also find that observing how children use physical resources gives an insight into their thinking. If a child is attempting to describe a transformation by trial and error and giving a muddled explanation, this would suggest they don't yet understand the activity. Plan to revisit it, focusing on careful use of mathematical language and imagery, perhaps using geoboards and rubber bands, or by moving cut-out shapes on coordinate grids.

Children self-correcting – that is, working by trial and



improvement rather than simply by trial and error – suggests their understanding is developing. Give them time to experiment and practise the activity and encourage them to discuss their ideas.

Children communicating clearly about what they have done, whether with apparatus, in conversation or on paper, suggests they have gained a solid understanding. Plan, then, to move them on.

What support is there for making summative assessments?

Assessment milestones and tracking children's progress

The medium-term plan in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* includes milestones – summary statements of specific points that children need to have a good understanding of before they move on to the next set of activity groups.

The milestones are based on the assessment opportunities in the preceding activity groups and are also aligned to the National Curriculum in England (2014). Your ongoing assessment of each child will build up over the preceding period and you can keep a record of attainment and track progress using the Numicon 6 Milestone Tracking spreadsheet, available to download from Oxford Owl, or the photocopy master of the milestones for the year (Photocopy master 1a–1b in the Teaching Resource Handbook).

Each milestone represents a point at which to reflect on each child's achievement and decide whether you need to plan further support and practice for them, giving them time to consolidate their understanding, or whether they are ready to move on. If children move on before they are ready, their difficulties are likely to be compounded, because they will not be adequately prepared for the new ideas they meet.

Explorer Progress Book

Each activity group has two corresponding pages in the Explorer Progress Book. One page generally poses a problem that challenges children to use the mathematics they have been learning in the activity group within a new context. The other page generally aims to provide more open opportunities for children, enabling you to assess their ability to think and communicate mathematically and also allowing you the opportunity to see the methods children use as they persist with an exploration.

The Explorer Progress Book is designed to be tackled in focus groups, so that you can administer and monitor each child's responses. In this way, you are able to build up a cumulative idea of a child's progress. The tasks enable you to assess children's ability to think mathematically and persist in their work, as well as whether they understand when to use particular mathematics skills. They are as open as possible, inviting a full range of responses. They are not pass or fail tests, rather they are here to support you in assessing as accurately as possible children's current understanding, so that you know what needs addressing. It may be useful to keep notes on children's responses and the significance of these responses for future work.

Consider carefully when to give children each Explorer Progress Book task. You might ask them to complete one page at the end of their work on the activity group and then the other two weeks later, to check how much they may have retained. Alternatively, you might give children both pages

after they have completed the next activity group, or just before they face the next related activity group. The aim is to gather information about children's understanding at a point when this information is useful for their learning – decide which is the most useful point, in each case.

Children should have available to them all the materials and imagery that have been available during the teaching of the activity group, and should be invited to express what they are doing as they do it. It is best to avoid affirming or denying anything a child says or does as they work; look and listen for what children do without your guidance.

Keeping track of what children remember and developing flexibility

Children's responses to questions and problems and their ability to give examples or make up their own questions in the 'morning maths meeting' or other practice sessions will indicate whether they are maintaining fluency with past learning.

You might choose an activity from the practice section of a completed activity group, vary the context and present it to children without preparing them in advance – notice what they do and do not seem to remember, and plan accordingly for the next related activity group. You can also use questions and activities from previous activity groups to help keep children's past learning and creative thinking 'simmering'.

What about formal testing for national authorities?

Formal tests and examinations are important hurdles for children and teachers, parents and carers, schools, universities, professionals, employers and governments. They also tend to be artificial settings in which to 'do' mathematics. In this sense a formal test does not correspond to children's encounters with mathematics in their learning or everyday lives, and this means devoting time to preparing them for the uniqueness of the experience.

In a formal mathematics test, communicating is almost always restricted to 'on paper' forms; this allows for some imagery, but not usually for action with physical materials. Also, the language used in test papers can be very formal. Thus children will need plenty of practice at interpreting such language and 'internalizing' their use of action and imagery. The development of mental imagery is a key aspect of Numicon, and children should be encouraged to 'imagine' actions, objects, movements and shapes as often as possible.

Children will also need to prepare for encountering difficulty in formal tests. In their mathematics lessons, they are encouraged to express difficulty – to explain why something is challenging and to use action and imagery to illustrate their thinking. Under exam conditions they will need to respond positively to being 'stuck' by communicating mathematically with themselves, working silently to express what the trouble is and using mental imagery to explore possible solutions.

Tests and examinations should not become the paradigm for 'doing mathematics', however. Children need to learn to function mathematically in a very wide range of situations. For ongoing assessment of children's understanding, allow them the full range of actions, imagery and conversation, and encourage them to communicate mathematically in their own way.

Key mathematical ideas: Geometry in the primary years

Underlying the activities in Geometry, Measurement and Statistics 6 are many key mathematical ideas that children will be developing and extending, as well as some conventions they may be meeting for the first time.

In order to teach these ideas effectively, those who are working on the activities with children will need to be clear themselves about the mathematical ideas and about which activities address which ideas.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the Geometry activity groups of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*. The educational context page of the activity groups lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to this section.

The mathematics coordinator may also find it useful to work on the key mathematical ideas in professional development sessions with the class teachers and the wider school staff.

Introduction

It is important to remember that doing geometry or measuring, and using statistics are significantly related aspects of mathematics, and that each of these areas of activity has strong connections with the other two. (It is important to remember too that all three also have strong connections with number, pattern, and calculating activity.)

'Geometry' in ancient Greek means 'earth measurement'; many physical measures involve geometrical ideas (e.g. pressure is usually measured per 'square' unit). Patterns in data (themselves usually records of measuring activity) are commonly represented visually – as are numbers themselves, of course, in the form of number lines.

Each of these areas of mathematical activity also has its own distinctive concerns: shapes are commonly classified with little or no reference to their size; we measure time, force, and temperature as well as distance, area, and volume; in statistics we are often concerned to interpret and illustrate a 'significance' in identified relationships. In this chapter we treat these three broad areas of mathematical activity relatively separately for ease of reference to their key ideas, but when you are teaching each area remember to exploit inter-connections at every opportunity.

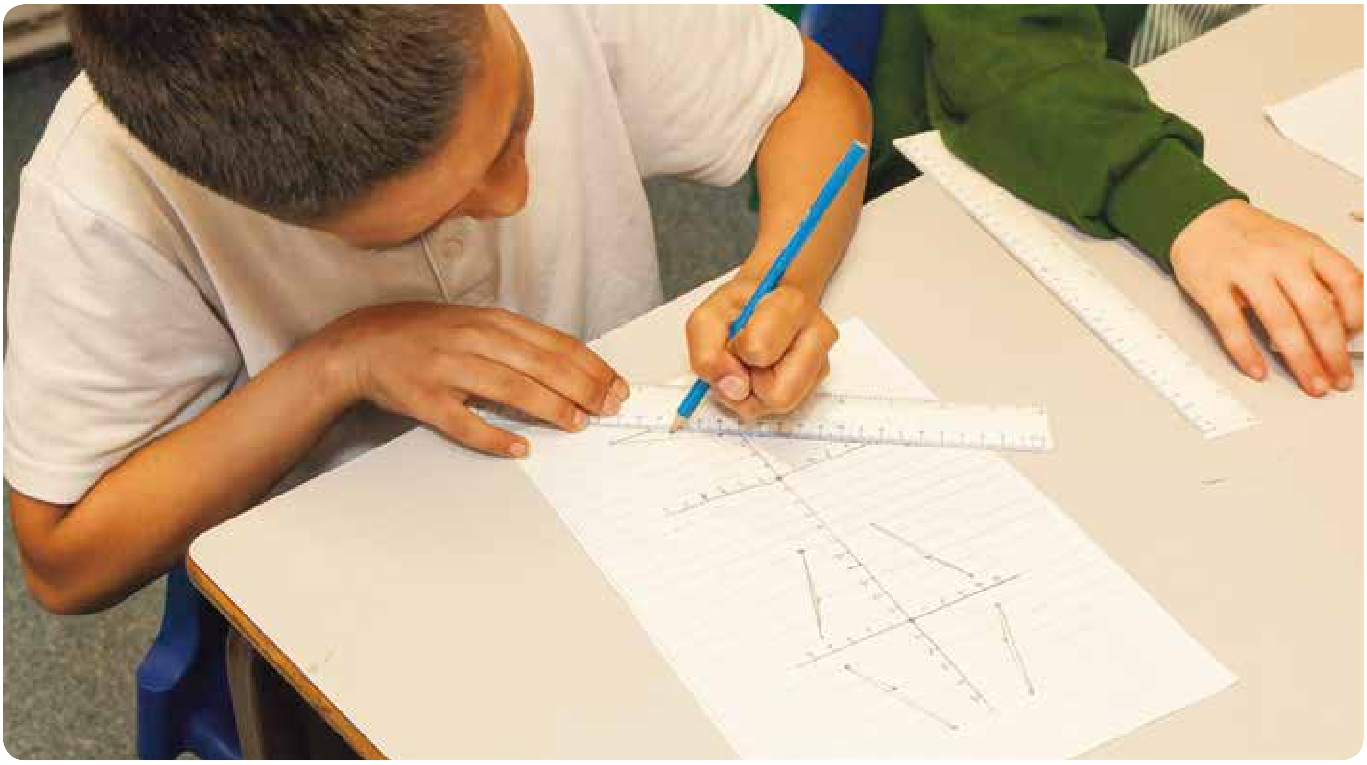
Geometry in the primary years

Children begin exploring shape and space literally as soon as they are born. In fact, it was Piaget's view that young children's first experiences of the physical world are closely related to the central ideas of a relatively recent geometry called *topology*. (In topology, key ideas include 'proximity', 'enclosure' and 'inter-connectedness', and it is easy to understand why such things are so important to infants from their earliest days.)[1]

In order to engage children in doing geometry at school, they need first-hand experiences with shapes and with position, direction and movement, within which shared opportunities arise both to describe actions, movements and observations freely and to introduce conventional mathematical names and terms. The point of using mathematically conventional language is to help develop 'a common way of seeing' a world with children, a common way of *communicating* about a geometrical world, so that we can share our individual perceptions and experiences with each other more easily. In the early years, children spend much of their time learning to join in with established geometrical conventions (ways of seeing and describing), as well as developing their individual 'seeing eye'. These two things need to proceed together, in close relation to each other, to allow children to make sense of the geometrical world.

The importance of first-hand practical experience for children is two-fold: young children do not yet think about shape and space in the same ways that adults do, and for

1 If you want an example of topology in adult use, think about a map of the London Underground and about why such ideas are the important ones in this context.



communicating to become effective in any context there has to be a shared underlying experience that all involved can build their communicating upon. (It is very difficult to communicate effectively about, say, a film that only one of you has seen; similarly, it is very difficult to *discuss* with children things that only you have experienced.) Children should be physically active and thoughtfully reflective in their geometry activities, and fully involved in *discussion* in all activities. It is only children's *agreement* to join in with doing mathematics and its established social conventions that will open up its possibilities to them. Children who are asked to use terms they can't relate to their personal experience – however important we might think those terms are – will find themselves easily forgetting those words.

Bear in mind that in learning to do geometry in school, children encounter some very old conventions traditionally used in classifying 'shapes' and describing 'position', the origins of which often lie thousands of years ago. It is quite easy for children sometimes to feel that they have to learn a strange foreign language to talk about abstract shapes in school, the point of which (at the time) can seem obscure. As with medicine and the law, many of the terms we use today in geometry have Greek and/or Latin origins and make good descriptive sense in those ancient languages, e.g. 'tri-angle' meaning three angles. Today, using words like 'polygon' and 'quadrilateral' can seem a bit odd. If children are introduced to the Latin and Greek roots of these terms (e.g. 'poly' meaning 'many', and 'quad' meaning 'four') they quickly get used to connecting the sense of these old words with what they are used to describe. This deeper level of reading can help develop their geometrical thinking and communicating significantly.

Doing geometry – transformations, invariants and equivalence

An important general point concerns the history of geometry and, consequently, what we tend to emphasize today in schooling. For many hundreds of years after the Greeks and Romans there was effectively only one kind of geometry used in western civilization – that of Euclid (fl. 300 BCE). But during the 19th century CE, mathematicians' attention began to focus on connecting new and ever more varied kinds of geometry with each other.

One key connecting idea turned out to be no longer thinking of geometry as simply 'earth measurement', but instead as the study of 'invariants under transformations'. People began to think of geometry as studying what changes and what stays the same, as various transformations (e.g. 'rotating', 'reflecting' or 'translating') are performed on shapes and space.

With the introduction of this idea, Euclid's traditional geometry has today become concerned with studying what stays the same (invariant) as we transform shapes. For example, a square will still have equal sides and four right angles, equal diagonals, the same area and so on, however it is rotated, reflected, or moved about. Conversely, modern topology is concerned with what stays the same under what are called 'rubber-sheet' transformations. Imagine what happens to shapes drawn on a rubber sheet as the sheet is pulled and stretched in any way that you like. A 'square' when stretched about on rubber will still form a continuous boundary (so keep an 'inside' and an 'outside') so we would say that 'enclosure' is invariant. Points that are close together originally will stay relatively close together, so

we say that 'proximity' is also an invariant under topological transformations. Actual measured distances and angles all change when shapes are stretched and pulled, and so they are not invariants.

Those topological invariants are the only ones that matter to a traveller on an Underground system, and also in young infants' spatial worlds. Studying 'invariants' under different kinds of transformations turns out to be a very useful way of understanding shape and space. This informs children's geometrical activity in school today: essentially we want children to *explore* 'What happens if we *do* this …?'

This in itself requires a necessarily *active* approach to doing geometry. We want children to be dynamically making shapes, and moving shapes (and themselves), and noticing and discussing what changes and what stays the same. 'Transforming' involves action, as does moving between and among 'positions', and *discussing* all this action is what allows children gradually to join in with the language and conventions of doing the geometry that we use in mathematics today. Gradually too, children will begin to learn how we *reason* about aspects of shapes and space in doing mathematics.

Equivalence

Just as children need to learn about equivalence in number work and algebra (e.g. $2 \times 3 = 6$, and $a + b = b + a$), so they will meet important equivalence relations in geometry. In primary school geometry, equivalence judgements are typically made in relation to specified transformations. Most commonly, children will learn that two shapes are 'the same as' each other if they could be rotated, reflected, and/or translated onto each other; two such shapes are called **congruent** to each other. If two shapes could be rotated, reflected, translated, and/or **scaled up or down** onto each other they are said to be **similar**.

Importantly, transformations can be performed one after the other, and some sequences of transformations are *equivalent* to each other in that shapes 'end up in the same state/place' after them. For example, a rotation of 180° about a point, followed by another rotation of 180° about that point, is *equivalent to* one single rotation of 360° about the point. Explicitly investigating sequences of transformations and their equivalences is a significant part of school geometry, though children will have already begun to explore this in their very early pre-school play with their simple handling of shapes (e.g. fitting different shapes into differently shaped holes).

Doing geometry – being logical

As in doing any kind of mathematics, when we do geometry we reason by making and using **generalizations**. Indeed, this is one key aspect that distinguishes doing geometry from measuring; when we measure, we are always measuring something specific, something particular.

Being logical in doing mathematics usually involves using what is called deductive logic; and *deducing* something involves moving from a general statement to a particular one.[2] For example, knowing that 'the exterior angles of any

2 There is a form of reasoning used in mathematics called 'proof by induction' because it moves from particular relations to a general conclusion about a whole sequence of relations. This form of proof nevertheless also relies upon at least one initial generalization about the sequence involved, for its validity.

polygon add up to 360°' allows us to deduce that the exterior angles of a triangle will add up to 360° because a triangle is a polygon. By using generalizations in various logical ways in mathematics, we can be sure that our reasoning is reliable.

As another example, we know that we could not try to tile a flat surface with regular pentagonal tiles because each interior angle of a regular pentagon is 108°, and there is no way you can fit a whole number of 108° angles together to make a 360° complete intersection (that is, leaving no gaps) because 360 ÷ 108 = 3·33, which is not a whole number. This is a **logical** argument: if the exterior angles of a regular pentagon add up to 360° (because it also is a polygon), then each exterior angle must be 72° (because 360 ÷ 5 = 72). If each exterior angle is 72°, then each interior angle must be 108° (because an interior and an exterior angle added together make a straight line, that is, 180°, and 180 – 72 = 108).

We have no choice about accepting reasoning like this. If our generalisations are valid our conclusion is *necessarily* correct. Notice just how much this reasoning depends upon using our *generalizations* (including numbers themselves), *logic* and *definitions* (e.g. of exterior angles). Doing geometry, that is, reasoning about shape and space relationships, depends crucially upon such logic, our definitions and generalizations.

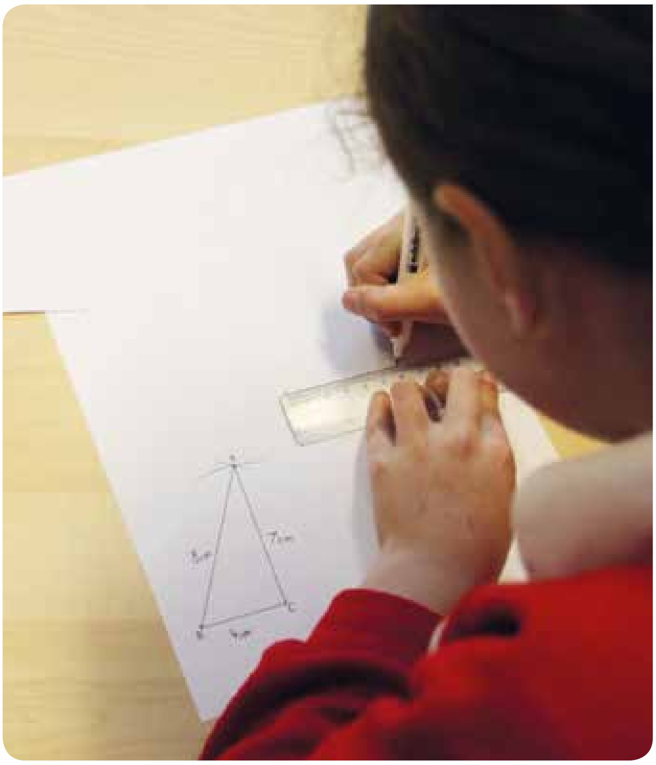
It is worth noting too, that though we may make, move and draw physical shapes on paper to help us to think, our logical reasoning is about shapes *in general*. Importantly, we can only contemplate such general shapes in our heads, that is, we can draw a particular triangle of particular dimensions physically but not a 'general' triangle. The general 'triangle' we reason about in geometry is *imagined*, and hence no measuring is involved.

Of course, as Piaget noted, young children are generally not yet capable of reasoning for themselves with the kind of 'formal operational' logic described above, but in primary school we can do much to prepare the way for children's mature geometrical thinking, and for their understanding of what 'doing geometry' involves. We can encourage children to *imagine* shapes and movements, we can encourage them to *generalize*, and we can encourage them to notice that there are significantly different ways of being 'correct' in mathematics, through ourselves always being careful how we answer the question, 'Why …?' We can also teach children what doing geometry is *not*.

What geometrical reasoning is *not*

It is quite common for children to want to test a generalization, such as 'the angles of any (flat) triangle add up to 180°', by drawing or choosing a particular triangle and measuring the angles with a protractor (or tearing off the corners of a particular paper triangle and re-arranging them in a line). There's possibly a feeling that lots of people could have done this in the past, and that every time anyone does it, they always come up with 180° – roughly. And so, if we could measure accurately enough, measuring lots of different kinds of particular triangles would eventually *prove* it.

This kind of activity is misleading for children in an important way. Working like this is how scientists work – by dealing with series of individual cases rather than reasoning about a *general* triangle, as mathematicians do. If all we did in geometry was measure actual particular triangles, however many triangles we measured practically – even assuming we could measure anything perfectly accurately (which we can't) – there would always be the possibility that in a triangle we



haven't looked at yet, the angles might come to more or less than 180° because we have no special *reason* for knowing it to be impossible.

In mathematics we reason *logically* about our generalizations using our imagination, and children will find that it is possible to reason logically that the angles of a flat triangle *must* add up to 180°, without doing any measuring at all.[3] (And further, that if by measuring we come up with 181°, then the extra 1° is a measure of the *inaccuracy* inherent in our measuring. It is our measuring that is wrong.)

As teachers, we don't help children understand what doing geometry is about if we encourage them to believe that the angles of a triangle really do add up to 180° because they've measured some triangles, or because they've torn the corners off one and arranged all three angles physically along a straight line. *Convincing* them is not the aim; *finding logical reasons* is.

Being logical is also not to be confused with being conventional

Just as children need to learn the difference between measuring and being logical, they need to learn the difference between being conventionally 'correct' and being logically 'correct'. Being conventional is a matter of social agreement; being logical leads to a *necessary* acceptance of truth (it is not a matter of any kind of agreement).

The fact that three-sided polygons are called 'triangles' is a *social* agreement (they might have been called 'trilaterals'); that the exterior angles of any polygon add up to 360° (that is, one whole revolution) is *necessarily* true – there is no choice about it.

(Have you tried drawing 'any polygon' yet, by the way? It is a two-dimensional closed shape, consisting of straight sides …)[4]

We learn conventional names for things in geometry in the same way that we learn conventional names for things in any other walk of life, simply by agreeing to go along with what everyone else seems to call things. This makes our communicating 'correct' in a social sense, but not in a logical or necessary sense.

We can help children learn this important distinction through the ways that we answer their questions. This in turn will help children learn that there are different *kinds* of 'facts' in geometry, and therefore importantly different ways of being 'correct'.

Because…

When children ask 'Why ...?', teachers need to make sure they give the right kind of answer. If a child asks, 'Why are there 360° in a whole turn?', we need to say something like, 'Well, the Babylonians used to love numbers like 60 that you can divide up exactly without using fractions, so they thought 360 (6 x 60) was a really helpful number for dividing up a circle'. In other words, it's a convention and one can point to a social history.

If a child asks, 'Why do the angles of a quadrilateral add up to 360°?' we need to invite them to *reason* it through. 'Because by drawing a diagonal you can divide 'any quadrilateral' (a generalization again, in your head) into two triangles, and we know that the angles of a triangle always add up to 180°, so 2 × 180° = 360°.' It's *logical*; we have no choice.

If a child asks (pointing to a shape), 'Why is that a triangle?' we need to be careful to give a full answer. 'It's called a triangle because we've agreed to put *all* shapes like that one (all those flat, closed shapes with three straight sides) together into one group, and call them all 'tri-angles' – probably because they all have three angles as well.' It's an *agreement* that could have been otherwise.

And so as children work with and explore shapes and movements, and speculate and try things out, they will find themselves both 'correct' and 'wrong' in significantly different kinds of ways. The ways we that use the word 'because…'

3 The exterior angles of any triangle add up to 360° (because it is a polygon). Also, the combined interior and exterior angles at each vertex of a triangle add up to 180° (they make a straight line), so the total of all six exterior and interior angles together for any triangle will be 540° (i.e. 3 × 180°). Since we know that the exterior angles together account for 360° of this total, then the total for the interior angles together *must* be 180° (i.e. 540° – 360°).

4 Trying to draw a *general* polygon illustrates very well how mathematical objects are works of the imagination. We can define it, describe several of its visual properties, and develop valid theories about it, but *draw* it we cannot.

will help them develop important distinctions. Children need to know *in which way* they are 'right' or 'wrong', and that in the world of shapes, as with people, 'a rose by any other name would smell as sweet'.

Parts, properties, movements and definitions – the social basis of geometry

Definitions are social agreements in the same way as the names of the categories they define. We first *agree to define* categories in a particular way, and then later use those categories in our classifying and reasoning. Doing mathematics is the shared occupation of a social community, and within mathematics definitions are *chosen* (and hence allowed to change).

We tend to use parts and properties that we distinguish within shapes and movements to define our conventional categories, and so early work with children involves inviting them first to agree to our *distinctions between* certain parts, properties and movements before doing any categorizing based on these. None of this yet involves logical necessity; it is still only about inviting social agreement.

Initially children seem to distinguish between shapes that are named for them in a *holistic* way, simply learning, '*That's* (called) a triangle' and '*That's* (called) a square' and so on, without distinguishing any parts.[5] Gradually their discrimination becomes finer and they are able to distinguish **parts** such as 'corners' and 'lines' that they can later agree to call 'vertices' and 'sides' and so on. Later still, children will begin to notice how parts relate to each other to give shapes **properties**, e.g. a trapezium has just two sides that are parallel to each other, or adjacent sides of a rectangle meet at 90°. Children also learn to distinguish **movements** from one another, e.g. between turning a shape around (rotating), and turning it over (reflecting).

So as young children make and handle shapes, fit them together, and move them (and themselves) around, they gradually learn to *join in* with the conventional language we offer them to describe the distinctions they see and notice with us. We invite their agreement to joining in with the language that reflects the distinctions and definitions the rest of us currently use. All of this is social activity – and the agreements could always be otherwise.[6]

Using transformations and invariants to name and define

By playing with, handling, and fitting together collections of physical shapes children will be using the Euclidean transformations of rotation, reflection and translation. This is a world in which lengths, angles and areas all stay the same (invariant, constant), and are therefore features *constantly available to be noticed* as shapes are moved around and fitted together.

5 See Van Hiele, P. N. (1986) *Structure and Insight. A Theory of Mathematics Education.* Orlando: Orlando Academic Press

6 To help children realize this, one very nice activity is to invite them to make as many different pentagonal shapes as they can, and then to sort these into different kinds of pentagon themselves, also inventing names for the categories they make. Since there is no conventional tradition for this (as there are for triangles and for quadrilaterals), the experience can illustrate very well to children the fact that names, categories and definitions of shapes are all simply social agreements – which could always be otherwise. (See David Fielker, (1981) *Removing the Shackles of Euclid* Derby: ATM.)

By including similar physical shapes as well as congruent ones in the collection, the transformation of **scaling** can be imagined. (Note the transformation of scaling affects a previous invariant: area.)

Under the transformations of rotation, reflection, translation, and scaling together, the angles of a square stay the same – they are invariant – as are the **ratios** of lengths to each other. Paying attention to just these invariants leads to a 'square' being defined in our geometry as a polygon with *four right angles* and *four equal sides*. These invariants can be used to define the shape. As the actual lengths of the equal sides and the area all change under scaling, we don't use the changing attributes of individual lengths or area, which vary, in defining a square.

Triangles are a different challenge to organize – there are many kinds of triangle, whereas there is only one kind of square. Children need to make and meet many different kinds of triangle before they can realize again that what stays invariant in any kind of triangle under rotation, reflection, translation and scaling are the angles and the ratios of sides.[7] They will notice that no one seems to care how long individual sides are, or how big the angles. As long as there are three of those things making this closed, flat shape everyone calls it a 'triangle'.

Significantly, noticing what stays the same and what changes as we transform something leads naturally to generalizing. In effect, children are learning to use terms such as 'triangle', 'square', 'oblong', 'circle' and so on, in ways that acknowledge that, e.g. 'a triangle' (that is, *any* triangle) is a closed shape with three straight sides forming three angles. These are the *invariants* of 'a triangle' in the geometry we begin with at school; any other properties that you might notice when you look at a particular triangle (e.g. this one's got a right angle) could *change*, yet the shape would still be a triangle. Similarly, it doesn't matter 'which way up' a triangle is (that is, how it is rotated), it will still be a triangle. Orientation can *change* – 'having a point at the top' is not an invariant property of triangles in the geometry we use.

It is clear how physical experiences with a wide range of *dynamic* materials are essential for children doing geometry. The active making, moving, describing, transforming and combining of physical shapes that children do with physical and IT materials forms the vital basis of their flexible *imagining* and forming of categories.

Gradually, through meeting, making and moving shapes in their activities and discussing what they and we see changing and staying the same, children learn to use words (category names) in the same ways that we do. These distinctions and agreed categories then become formalized into definitions.

Definitions essentially spell out explicitly the agreed boundaries around a category. They are a kind of *verbal contract* underlying discussion that is always open to revision and scrutiny. 'Triangles' become defined as 'closed, flat

7 They are unlikely to be realizing this consciously, or be capable of expressing what they see in these terms. What they are *noticing* – informally – is that the 'corners' don't change however you move the shape about, and that it still *looks as if* it's the same shape (it's what we call 'similar') whenever we do those things with it.

shapes having exactly three straight sides' for as long as it suits us to look at them that way.[8]

In our **Glossary** we list the definitions that are usually agreed for work in primary schools. Notice how important your visual imagination is to you as you read these agreements, and remind yourself just how important children's physical experiences with actions and movements are to them, as you discuss and invite them to agree with the conventions you suggest.

From defining towards classifying and relating

It is one thing to be able to distinguish and name shapes (and parts and properties of shapes), but it is quite another to be able to relate such categories to each other and to reason with such relations. Being clear about categories is fundamental to thinking logically with them.

In schools in some countries much early discussion with children focuses explicitly on 'different *kinds* of things' (or categories), relations between categories, and hierarchies of categories, so that children's attention is consciously drawn to the importance of category distinctions for logical thinking. This becomes important with children doing geometry as we introduce them to relations between categories of shapes and the ways in which inclusive hierarchies of categories lead to the possibilities of reasoning with ever broader generalisations.[9] Putting things into categories and relating those categories to each other is called *classifying*, and classifying shapes is one key way in which we put a conventional order and structure onto the many possible worlds of geometries.

Sometimes category distinctions are very subtle. *Logically* we can't add different *kinds* of things together; '3 m 15 cm' is not 18 anythings, it's *either* 315 cm or 3·15 m. Children will encounter this logical feature again when they try adding together, for example, $\frac{1}{4}$ and $\frac{2}{3}$. Conventionally however, we often say '3 metres 15 centimetres' as if we've 'added' them together, whereas we've simply put them *next to* each other, one after the other, not combined them.

And so to generalizing in geometry …

In doing geometry, having agreed categories and definitions, we then reason about mathematical objects and their relationships *in general*, such as 'a polygon', 'a quadrilateral', 'a cylinder', or 'a sphere', and these generalized objects are similar kinds of things to those other mathematical objects we ask children to make – pure numbers, such as '6', '23', or '¯0·5'. As in number and algebra, in doing geometry we *reason* with our *generalizations* (whereas in measuring we're always measuring something specific, something particular).

What children importantly become better at as they work and develop through their primary years is the *virtual* action essential to generalizing. They gradually become able to reason about '*any* quadrilateral' and so on, and it is only possible to do this in our *imaginations.* Any quadrilateral that we actually make or draw is always a particular one; the important *general* quadrilateral that we want children to be

8 If we move to another kind of geometry, perhaps with shapes on curved surfaces, we might want to change our definition of 'triangle'.

9 E.g. polygon → quadrilateral → rectangle → square, is one such inclusive hierarchy of categories, enabling us to *deduce* logically that the sum of the exterior angles of a square will add up to 360°, because those of *all* polygons do, and a square is one kind of polygon.

able to reason about can only be constructed in our heads. The same goes for all the other geometrical generalizations we want children to make. In this important sense, geometry can only be done in our heads.

Notice that it's more difficult to imagine a general triangle, or quadrilateral, or polygon, or prism, than it is to imagine a general square. That's because there are many different *kinds* of triangle, quadrilateral, and so on, but only one kind of square (in our conventional classification). Again in this, the variety and range of children's physical actions, and the maximum use of *dynamic* materials (such as geo strips) is essential. The more *transforming* children can do physically as they notice and discuss what changes and what stays the same, the better.

When doing practical geometry with children always encourage them towards generalizing with questions like, 'Will that *always* work ...?', and 'What if ...?'. And encourage their reasoning with questions such as, 'Is that because ...?', and 'Why is that, I wonder ...?' Don't feel that some children are too young to be able to contemplate such questions; even before children are capable of answering them, the *habit of asking* such questions in one's teaching is what is important.[10] It is for the children to do the generalizing, the reasoning and the imagining.

Working in 2D and 3D

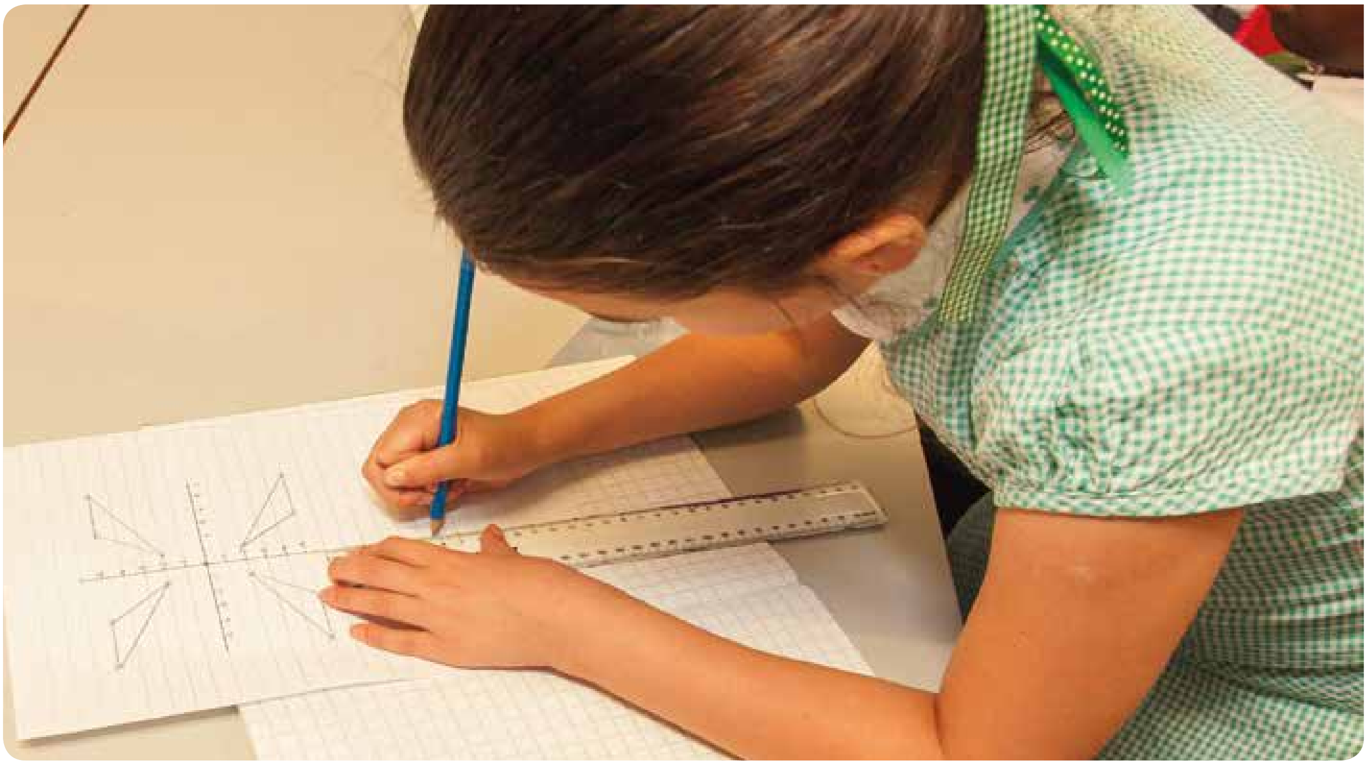
Essentially, working with shapes and space in three dimensions rather than two involves no change in approach. Work in 3D begins with children handling, rotating, reflecting and translating shapes in various ways, and also noting that scaling only changes lengths, surface areas and volumes. A cube is a cube, is a cube, however big or small it is. Angles are invariant under these four kinds of transformation, as are ratios of lengths. Fitting 3D shapes together is also a key explorative activity.

Some agreements on names are changed in a shift from 2D to 3D work, and some interpretations of transformations shift up a dimension. So for example, 'sides' become 'edges' in 3D, a 'rotation' would be around a line (instead of a point) and a 'reflection' would be around a plane (instead of around a line). Curved surfaces become notable in 3D, as opposed to curved lines in 2D. These are all conventional agreements we invite children to join in with, as their experiences in 3D invite new *discriminations between* **parts** and **properties**, prior to agreeing to use these discriminations in new definitions.

Once 3D categories are agreed (e.g. 'polyhedra', 'prisms'), children can begin to relate categories to each other, to classify, to *generalize* and to reason *logically* about aspects of three-dimensional space. Reasoning about the possibilities of tiling in two dimensions, for example, shifts up to reasoning about the possibilities of packing in three dimensions.

Activities directly connecting 2D and 3D parts, properties, transformations and shapes are invaluable. So in the early years, printing with 3D shapes offers important connections between 2D shapes and 3D 'faces', and later on, work on the 'nets' of 3D shapes does the same.

10 'The palace of reason has to be entered by the courtyard of habit.' See Peters, R. S. (1966) *Ethics and Education*. London: George, Allen & Unwin p314. For Peters, this was what he called 'the paradox of education'.



Essentially, work in 3D involves the same developmental sequence of activities as work in 2D: *discriminating between* parts, properties and transformations in order to agree conventional *definitions*; *generalizing* from such definitions; *imagining* 'a (general) cylinder', 'a prism' and so on, and thence to *reasoning logically* about such mathematical objects.

Walking the line… a difference of perspective

Finally, much geometry activity is done from the point of view of 'looking *at*' shapes, and moving them around in front of us. It is quite often very helpful however to shift perspective and to imagine walking around, along, and inside shapes. Young children do this readily and actually in climbing frames and playground equipment, but often, in school, activities are done on tables with materials in their hands, which almost exclusively involves 'looking *at*'.

Programmable robots and the programming language 'Logo', however, allow us to explore shape properties, and position and direction in ways that invoke the 'walking along and inside shapes' perspective to good effect. A shift in perspective often makes different things clear, and familiar things look helpfully different.

When children are 'looking *at*' polygons, for example, the only angles that are obvious are the interior angles. When they try making a polygon with a Logo turtle however, all of a sudden it's the unwritten exterior angles that become important – and the 'total turtle trip' experience travelling around such a shape allows children to appreciate how *all* the exterior angles *must* add up to 360°, in a very different, and immediate, way.

Encourage children to shift perspectives frequently, and so to unite experience of 'position' and 'direction' closely with what are often thought of as the more 'static' aspects of shape.

Doing geometry – using Cartesian coordinates

René Descartes (1596–1650) is generally credited with inventing a system for describing a position in space using axes and a set of numbers, called coordinates. This revolutionary idea subsequently allowed algebra and geometry to become united so that equations could be interpreted visually, and shapes could be defined and explored with algebraic equations.

Doing geometry with coordinates – uniting geometry with algebra – is usually called 'analytic geometry', and René Descartes is its father. Analytic geometry is developed systematically in secondary school mathematics, but children first met this approach as they were introduced to Cartesian coordinates in Geometry, Measurement and Statistics 5.

Geometry in Geometry, Measurement and Statistics 6

Overall, Geometry, Measurement and Statistics 6 sees an ever-closer integration of geometry, measuring and statistics work. For example, children construct shapes for themselves using rulers, compasses and protractors, and develop work on the visual illustration of relationships with linear graphs and pie charts, while extending and deepening their understanding of forms of graphical representation.

There is a new emphasis in Geometry on making shapes, both 2D and 3D, and this connects closely with children's developing skills of practical measuring. Constructing shapes, both drawing 2D shapes and making nets for 3D shapes, also usefully ties in with children's developing understanding of the properties of shapes, and their reasoning and calculating of unknown angles in triangles, quadrilaterals and regular polygons. 'Vertically opposite' angles are also introduced.

There is a similarly new emphasis upon properties of circles, including the important relationships between radius, diameter and circumference. Children are prepared for the introduction of 'π' by investigating the constant relationship between diameter and circumference, but do not yet begin working with this famously mysterious number.

There is a major development of work on Cartesian coordinates as both horizontal and vertical axes are extended using negative numbers to create the full Cartesian coordinate grid of four quadrants. Translations and reflections of simple shapes, and corresponding patterns in coordinates, are explored as children move yet further towards the 'algebra' of analytic geometry.

Scaling is further explored as children's work on ratio develops during the year, and they deepen their investigating of both invariance and of what changes under this non-rigid transformation. In particular, children begin to study how length, area and volume change at different rates as they enlarge and reduce geometrical objects in proportion to each other.

The transformations

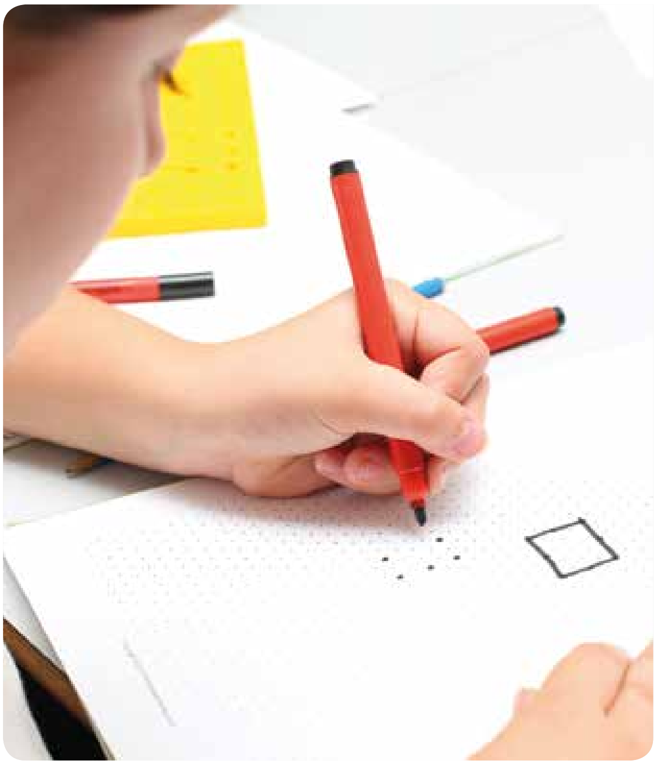
Children continue to explore and use reflections, translations and scaling formally in Geometry, Measurement and Statistics 6, but work on rotation is largely informal at this stage.

Rotation: Children develop their work with angles (as amounts of turn) both with increasing use of protractors and in noting the equivalence of 'vertically opposite' angles for the first time. Possibly the easiest way to visualize and confirm the equality of vertically opposite angles is to imagine the rotation of one straight line in relation to another, about their point of intersection (see the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, Geometry 1).

More informally, in exploring the possibilities of tangrams (Measurement 2) and in systematically devising different 2D nets for 3D shapes (Measurement 3), much practical rotating of shapes is necessary in order to establish the equivalence, or not, of various arrangements.

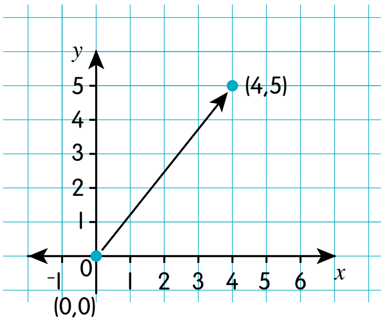
Reflection: Children develop their work on reflections further in Geometry, Measurement and Statistics 6 using Cartesian coordinates. At this stage they consider only reflections in the x- and y-axes in order to reason more clearly about corresponding patterns in changes of coordinates, e.g. reflecting a point (4,3) about the vertical y-axis will result in the point (–4,3). This kind of activity prepares children for later algebraic work on transformations with matrices in secondary school.

As with rotations, children will also be using reflections more informally as they explore the possibilities of tangrams, and as they try to establish the equivalence (or not) of various arrangements of 2D shapes with respect to forming nets of 3D shapes.



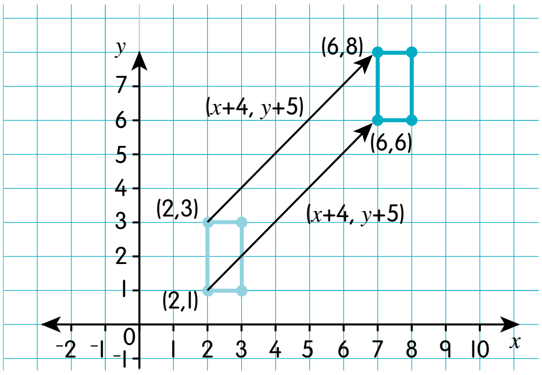
Translation: In Geometry, Measurement and Statistics 6, translations are again explored in the context of Cartesian coordinates (Geometry 3), and there is a continuing need to avoid possible confusion when using coordinates to describe a translation.

A translation is defined as a *single* movement of a particular distance in *one* direction – which is technically called a 'displacement', or a movement *directly* between two positions. Thus, using coordinates, we can specify a translation of a point by naming the coordinates of the beginning and end points, e.g. a translation from (0,0) to (4,5), and noting that the translation is a movement *directly* between these two positions.



However, in practice when using coordinates to find the end points of a translation moving a whole shape, children are often encouraged to describe this almost as if the translation were *two* translations, that is, 'four along' *and* 'five up' *for every point of the shape*, so that they can generalize the translation for all the points of the shape expressed in coordinates as '$x + 4$' and '$y + 5$', or:

$$(x,y) \rightarrow (x + 4, y + 5)$$



The valuable thing for children to learn with this algebraic move (the generalizing that addresses translating *every* point of a shape) is that apparently performing two translations ('four along' followed by 'five up') is *equivalent in outcome* to performing just the single direct translation that moves *directly* from starting position to finishing position, but it is not exactly the same thing.

It is important for children to realize that any translation actually involves travelling 'as the crow flies' directly between two positions, and not in two consecutive steps parallel to the x- and y-axes respectively, even though when 'talking coordinates' we might tend to say 'each point goes 4 along and 5 up'.

This may seem fussy, but it is important for children's later work that they realize every translation has just *one direction* and that it is always a movement along the line of the shortest distance between beginning and end positions. Technically, every translation is what is called a vector; it has a single magnitude and a single direction.

Again, children will be using translations informally as they explore the possibilities of tangrams and of 2D nets for 3D shapes.

Scaling: Children began to explore scaling explicitly in Geometry, Measurement and Statistics 5 with the introduction of scale drawings, and began to notice then that unlike the other three (rigid) transformations, some important properties *change* during scaling. In particular, lengths and areas change when a shape is scaled, but angles and the *proportions* of shapes do not, that is, angles and proportions remain invariant. Scaling was thus connected to the idea of 'similar' shapes: bigger and smaller versions of 'the same shape'.

In Geometry, Measurement and Statistics 6, Measurement 4, this work is extended to include the scaling of 3D shapes and children now begin to notice that – as was the case with lengths and areas – volumes also change when a 3D object is scaled up or down, although (again) proportions do not change.

Work on scaling thus draws crucially upon the ideas of ratio and proportion that children are meeting with their work on fractions, ratios and percentages in Number, Pattern and Calculating 6, and in Geometry, Measurement and

Statistics 6 children bring all these ideas together when exploring how shapes 'grow' in size. Their exploration involves use of the term 'scale factor' to investigate just how lengths, areas, and volumes all change at different rates to each other as shapes are enlarged or reduced.

Appropriately, concurrent work on circles in Geometry, Measurement and Statistics 6 focuses on how, as circles are enlarged or reduced, lengths and areas change, but proportions do not (Geometry 2). The ratio of circumference to diameter (circumference is 'three and a bit' times the size of diameter) is constant for all circles, however large or small. Circles are all 'similar' to each other, as are all regular shapes such as squares.

The invariants

The invariants children are concerned with in Geometry, Measurement and Statistics 6 are the properties of shapes that remain invariant, that is, they don't change under any of the above four transformations (e.g. angles). Probably the key development at this stage is children's exploration of how the *proportions* of shapes stay the same under scaling transformations. This helps to refine children's understanding of what a shape is as they look more closely at what happens as shapes are enlarged or reduced in size. Proportions are specified in terms of constant ratios.

It is important for children to be aware that scaling is a different kind of transformation to the other three; rotation, reflection, and translation are all called 'rigid' transformations since both the shape *and* size of the original stay the same when they are performed. Scaling is not a rigid transformation since sizes change, but not shapes.

Consequently, when drawing children's attention to changes in lengths, areas and volumes as shapes are enlarged or reduced, we are not focusing only on invariants but also on properties that *do* change. Scaling is a transformation that essentially involves a change of size in the shape itself.

In looking at how the *proportions* of a shape are retained under scaling, children's attention is drawn to how, although lengths (for example) change when a shape is scaled, the lengths all change in the same *ratio* as each other: if the length of an oblong doubles under scaling, then so will its width. In other words *all* lengths of a shape are scaled up or down in the same ratio, or by the same scale factor.

Another way of putting this is to say that if (say) a cuboid is enlarged or reduced, then the *ratio* of its height to its width will remain the same, that is, invariant. Similarly, if an oblong is enlarged or reduced, the ratio of its length to its width will stay the same. This is why all technically 'similar' shapes 'look the same'.

Furthermore, through this connection children develop their study of circles in Geometry, Measurement and Statistics 6 to begin to explore the important constant ratios involved: the ratios of radius to diameter, and of diameter to circumference. As circles are enlarged or reduced in size, these ratios remain constant – invariant – since all circles are 'similar' to each other, or 'the same (circular) shape' (Geometry 2).

The communicating

Children's mathematical communicating develops further in Geometry, Measurement and Statistics 6, through increasing use of Cartesian coordinates as the preferred method for communicating changes in position under reflections and translations, through children's increasing use of the term 'scale factor' when investigating scaling, and through their use of new conventional terms for describing attributes of circles. Importantly, in all these aspects children are developing their use of the major and conventional ways in which shape, transformation, position and direction are communicated very precisely when doing mathematics. Instead of just saying two shapes are 'similar', children can now point to a scale factor that specifies exactly the degree to which a shape is either larger or smaller; instead of just saying two shapes 'look the same', children can specify that the ratios of lengths (e.g. diameter to circumference) remain constant in similar shapes.

It should be noted that again (as in Geometry, Measurement and Statistics 5), the key aspect developing children's thinking and communicating is an increasing tendency towards using numbers, and hence algebraic language, when describing shapes and their changes. Using coordinates and scale factors to specify positions and enlargements or reductions introduces numbers more fully into children's geometry, and where there are numbers it is also possible to use algebra. Using algebra will give children the freedom to use the techniques they have available in algebra to *generalize* in geometry. Using numbers and algebra to do geometry is usually called 'analytic geometry'.

All of these developments in children's communicating and thinking are important for their future mathematical progress, and therefore require plenty of discussion and illustration as they are introduced. It is through *using* these new ways of communicating for themselves that children will gradually incorporate them into their own ways of thinking and communicating.

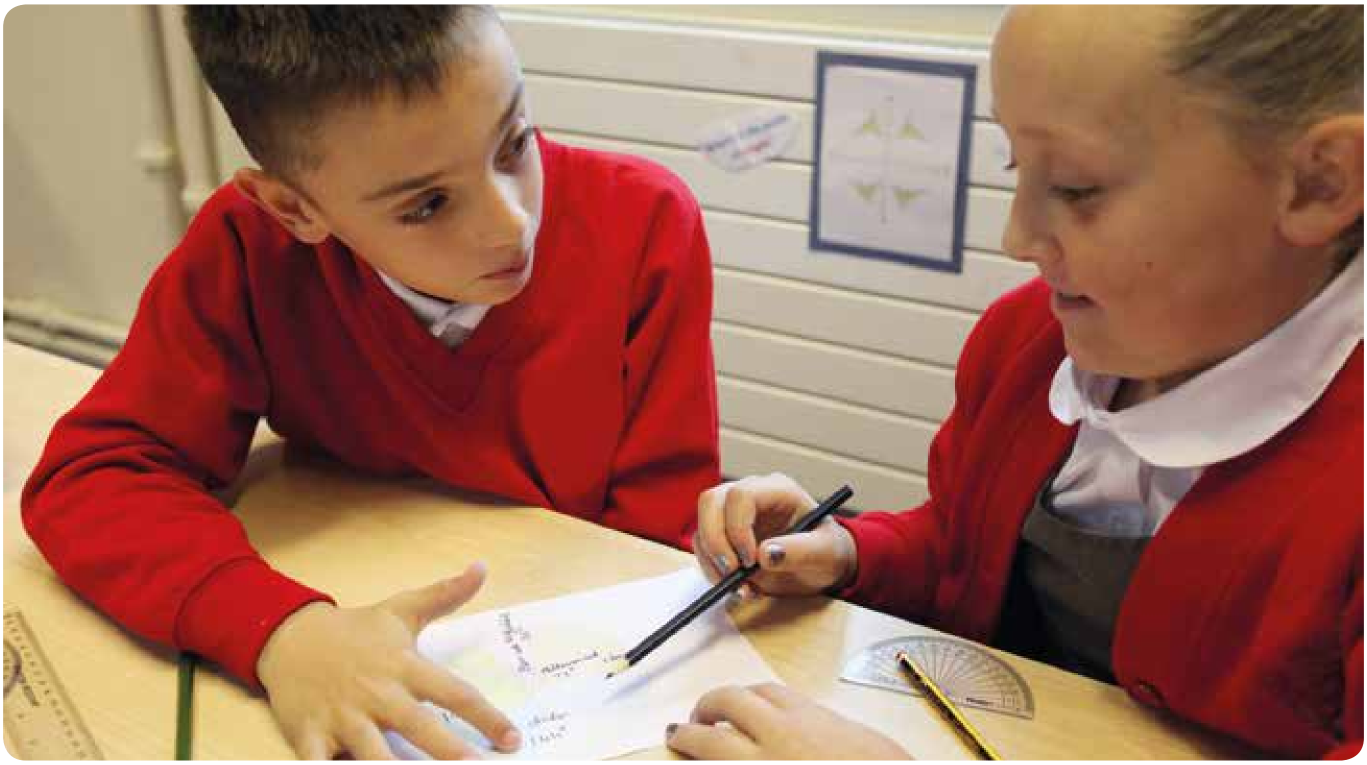
Key mathematical ideas: Measurement and Statistics in the primary years

Underlying the activities in Geometry, Measurement and Statistics 6 are many key mathematical ideas that children will be developing and extending, as well as some conventions they may be meeting for the first time.

In order to teach these ideas effectively, those who are working on the activities with children will need to be clear themselves about the mathematical ideas and about which activities address which ideas.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the Measurement activity groups of the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*. The educational context page of the activity groups lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to this section.

The mathematics coordinator may also find it useful to work on the key mathematical ideas in professional development sessions with the class teachers and the wider school staff.



Introduction

It is important to remember that doing geometry or measuring, and using statistics are significantly related aspects of mathematics, and that each of these areas of activity has strong connections with the other two. (It is important to remember too that all three also have strong connections with number, pattern, and calculating activity.)

'Geometry' in ancient Greek means 'earth measurement'; many physical measures involve geometrical ideas (e.g. pressure is usually measured per 'square' unit). Patterns in data (themselves usually records of measuring activity) are commonly represented visually – as are numbers themselves, of course, in the form of number lines.

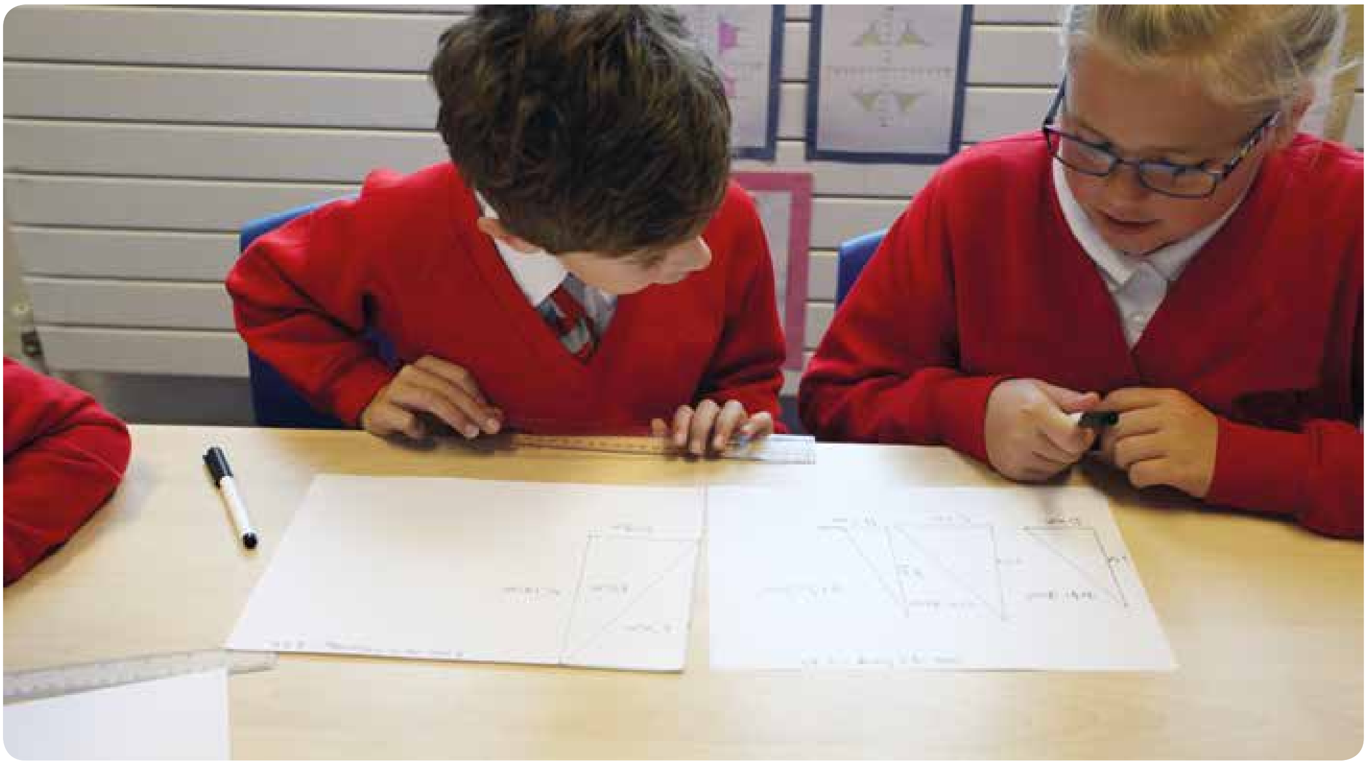
Each of these areas of mathematical activity also has its own distinctive concerns: shapes are commonly classified with little reference to their size; we measure time, force, and temperature as well as distance, area, and volume; in statistics we are often concerned to interpret and illustrate a 'significance' in identified relationships. In this chapter we treat these three broad areas of mathematical activity relatively separately for ease of reference to their key ideas, but when you are teaching each area remember to exploit inter-connections at every opportunity.

Measurement and statistics in the primary years

We always measure for a *purpose*, that is, measuring something is never an end in itself. Because of this, children's experiences with measuring are most effective when set within purposeful contexts. For example, when reviewing an arrangement of classroom furniture we could ask, 'I wonder if that bookcase would fit in that gap over there?' Similarly, measuring mass/weight, volume and capacity have a clear purpose when cooking, or in any other situation where we might want to be able to repeat something. Timing is important for organizing deadlines and appointments. Cross-curricular links generally are invaluable for offering great variety of purpose to children's measuring activity.

There is another consequence of measuring being purposeful that can be overlooked in primary school activities: our measuring in life is always only as formal or informal, as accurate or approximate, as suits the particular measuring *purpose*, on any occasion. As adults, we do not usually measure our drink of coffee at home in millilitres, nor do we measure how far apart to put our coats down in metres when making a 'goal' for a game in the park. Standard units do not become appropriate to measuring tasks simply because children are of a certain age, or because they appear from some point onwards in a curriculum. They are not somehow 'grown-up' or 'proper' measuring and should not be introduced as such.

Standard units are used only on particular occasions *for particular reasons*, usually to ensure clear communication, and/or when there is a lack of personal trust. When trying out a new recipe we usually measure the named amounts precisely, because we don't yet trust our own judgement in the new venture. Standard measures are very important



agreements in global trade and in scientific communication – both contexts in which 'trust' depends crucially upon careful use of agreed units. (We might not use standard measures for our drinks at home, but we certainly expect them in shops.)

If children are to fully understand measuring, the aspect of *purpose*, and its consequences, needs to feature clearly in all work and discussions. Standard units are not the 'best' units, nor are they the 'most accurate'. They are simply the units most appropriate to particular occasions. Probably we do more measuring in everyday life without standard units than with; children need to recognize why they use the types of units they do when they are measuring in all types of situations. Standard units are necessary only for *communicating* and for supporting *trust*. Children need to learn *when* standard units are appropriate, in context.

Measuring – contrasting and comparing

There is a sense in which all measuring can be understood as comparing, in that an underlying purpose is always to compare **qualities** or **quantities**. Interestingly, not all comparing involves measuring – sometimes we count, and sometimes we order, to compare, for example, the number of votes in an election, or the winners of a race.

It is worth noting that we cannot do any comparing until we have first distinguished whatever quality is to be compared. So contrasting qualities (distinguishing between them) is an essential prerequisite to comparing. This means that sorting, and distinguishing between, in topics such as length and capacity, different kinds of 'big' and 'small', are crucial preparatory activities with young children.

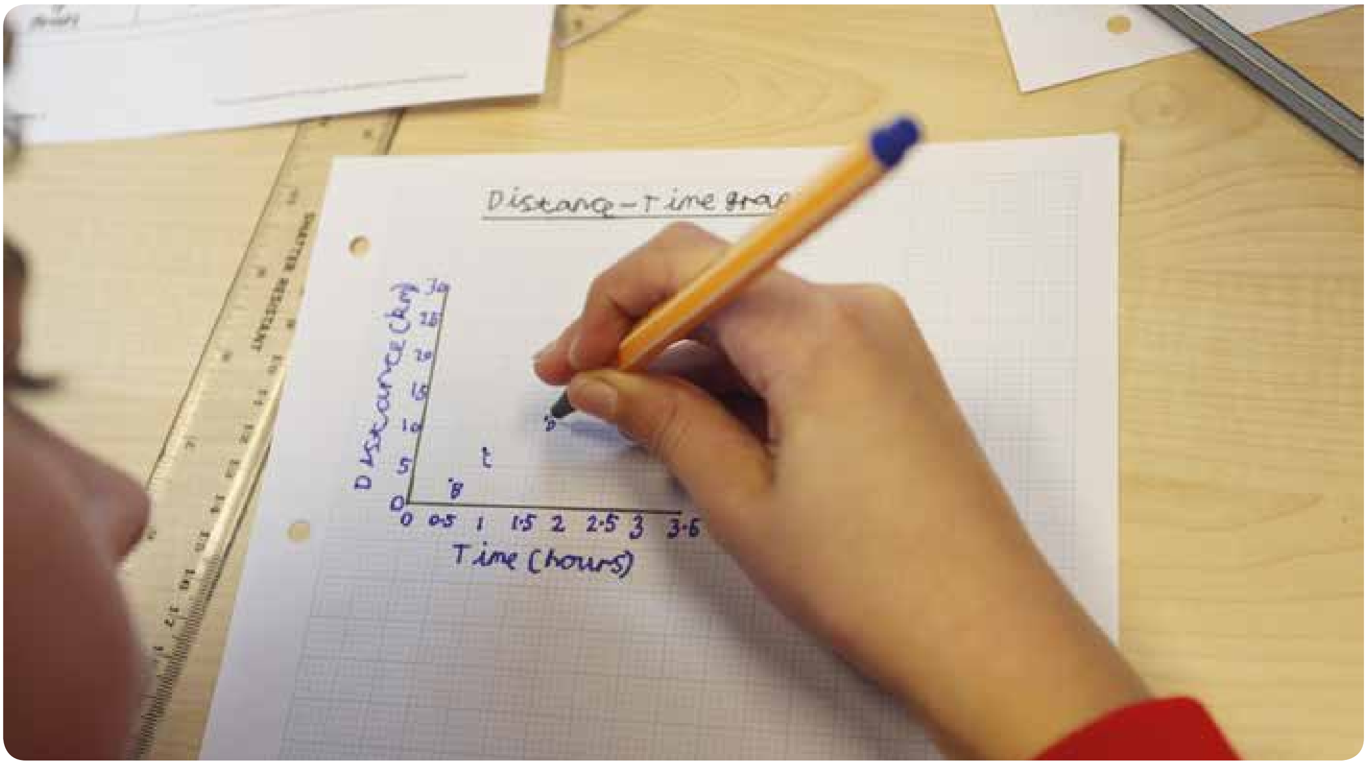
Measuring and counting – continuous and discrete quantities

There is an important difference between the two questions, 'how much?' and 'how many?' To answer the first, we measure; to answer the second, we count. **We measure continuous amounts, but we count discrete objects.** We measure time, length, area, volume and so on, but we count votes, cars and the number of items in our basket.

This distinction is important – where measuring involves a continuous quantity, the outcome is always an *approximate* figure. This observation itself has important implications for our use of continuous measuring scales and instruments, and for always deciding on an appropriate level of accuracy in any situation. It also has very important connections with the work we do with children on fractions and interpreting the spaces between whole numbers on a number line.

There are a variety of questions that ask about 'how much' of something we have, and so lead us to measure: 'how far' and 'how long' ask about length, or distance, or time; 'how heavy' asks about weight or mass. 'Where is ...?' is a question that can invite a combination of measures, e.g. an angle and a distance. Volume, capacity and mass all tend to be asked about with the broad question itself, 'How much?' (e.g. 'How much sugar would you like in that?').

Children thus need to learn to use a wide variety of terms and language in association with purposeful measuring and comparing, including the usual grammatical distinction between having 'less' of something continuous, or having 'fewer' discrete things.



Measuring and types of scales

There are important distinctions between different types of measuring scales. These have implications for the kinds of calculating we can do with measurements we have made, and thus for the types of statistics we can do with measures and counts of various kinds.

As a handy reference, in 1946 S. S. Stevens proposed a classification of measures that has proved very productive in stimulating debate and that also connects usefully with our approaches to data handling and measuring.[11] It is worth considering, not only because of the connection with statistics, but because it explains why, although 20 cm is 'twice as long' as 10 cm, 20 °C is not 'twice as hot' as 10 °C (that is, we can't do the same calculations with temperature readings on the Celsius scale that we can do with lengths). Stevens' classification is not put forward here as a 'correct' view of measuring, but simply as one that has an interesting and helpful relevance to the development of both measuring and statistics work with children.

As a psychologist, Stevens was much concerned with statistics. His classification is therefore concerned with both measuring and handling data in the contexts of physical and social science. In statistics, both measuring and counting are used in contrasting (that is, distinguishing between things) and comparing.

11 Stevens, S. S. (1946) *On the theory of scales of measurement,* Science, 103 (2684): 677–680

The Stevens (1946) classification distinguishes four types of 'measuring' scale:

- **Nominal**: this involves simply distinguishing between (contrasting), and putting items into different categories, according to names or specific qualities, e.g. distinguishing between male/female, distinguishing between English/French/Spanish, and so on. Distinguishing between team players by giving them different numbers on their shirts is a kind of nominal scale. Importantly, no ordering or value is involved or implied.
- **Ordinal**: an ordinal scale distinguishes between and also *ranks* items qualitatively, but without attending to any degree of difference between ranks. This scale simply compares qualities of objects or events by position in an order. The Beaufort scale of wind strength is one example, as is the Mohs scale of mineral hardness. Social surveys often use ordinal scales when asking people if they 'agree strongly', 'agree', 'don't mind', 'disagree', 'disagree strongly', and so on.
- **Interval**: an interval scale discriminates between values, and orders them, but also focuses on constant *degrees* of difference between values on the scale. Examples are temperatures in °C, or dates on an infinite timeline. The intervals on the scale, e.g. in degrees or years, are all the same as each other, so *differences between* temperatures or dates can be computed and added or subtracted (and ratios *of differences* make sense). However, we can't sensibly 'add' or 'multiply' with dates or temperatures themselves. On 1 January 2010 we could say that 1 January 2000 was 'twice *as long ago*' as 1 January 2005 (that is, a ratio of differences between dates), but we cannot compare two dates (or temperatures) directly themselves

with each other in any other way apart from simply ordering them and saying how far apart they are.[12]

- **Ratio**: a ratio scale distinguishes 'difference', 'order' and 'degrees of difference', but also includes an important, non-negotiable 'zero' boundary that actually leads to different possibilities of calculating.[13] The classic examples of ratio scales are our physical measures of length, area, volume, mass and so on, where it makes no sense to think of 'negative' lengths, areas, and so on.[14] Importantly, this also means that differences between values on a ratio scale are also values themselves, and we can thus make sense of *ratios* of particular *values* to each other, e.g. a difference between two lengths is itself a length (whereas a difference between two dates on an interval scale is not itself a 'date'). We can compare two different values on a ratio scale in *two* ways: we can say that 6 cm is 4 cm longer than 2 cm (their difference), but we can also say that 6 cm is three times as long as 2 cm (their ratio).

This classification might seem a rather complicated background to work in primary schools, but it can actually help draw attention to several key ideas in measuring and comparing, and is highly relevant to the kinds of questions and data handling with which many primary school children can engage.

Notice that Stevens' classification is cumulative. First we pay attention only to *differences* between items and we simply *distinguish between* various qualities. This makes categories. This 'nominal' scale relates closely to children's early sorting activities, to their distinguishing of qualities such as 'heaviness' and 'volume' as refinements from their early global use of terms such as 'big' and 'small'.

Secondly, by using an 'ordinal' scale we still distinguish between items and qualities but we now *order* these categories or qualities as well. This relates to children's early comparing and ordering of lengths, heaviness, volumes, and so on.

Thirdly, the idea of repeated equal *units* and specified *values* on an 'interval' scale is introduced, and the combining of (adding) and finding differences between (subtracting) values becomes possible. This relates to children meeting the idea of measuring 'units', adding and comparing lengths and so on, and introduces the possibility of meaningful 'negative' values.

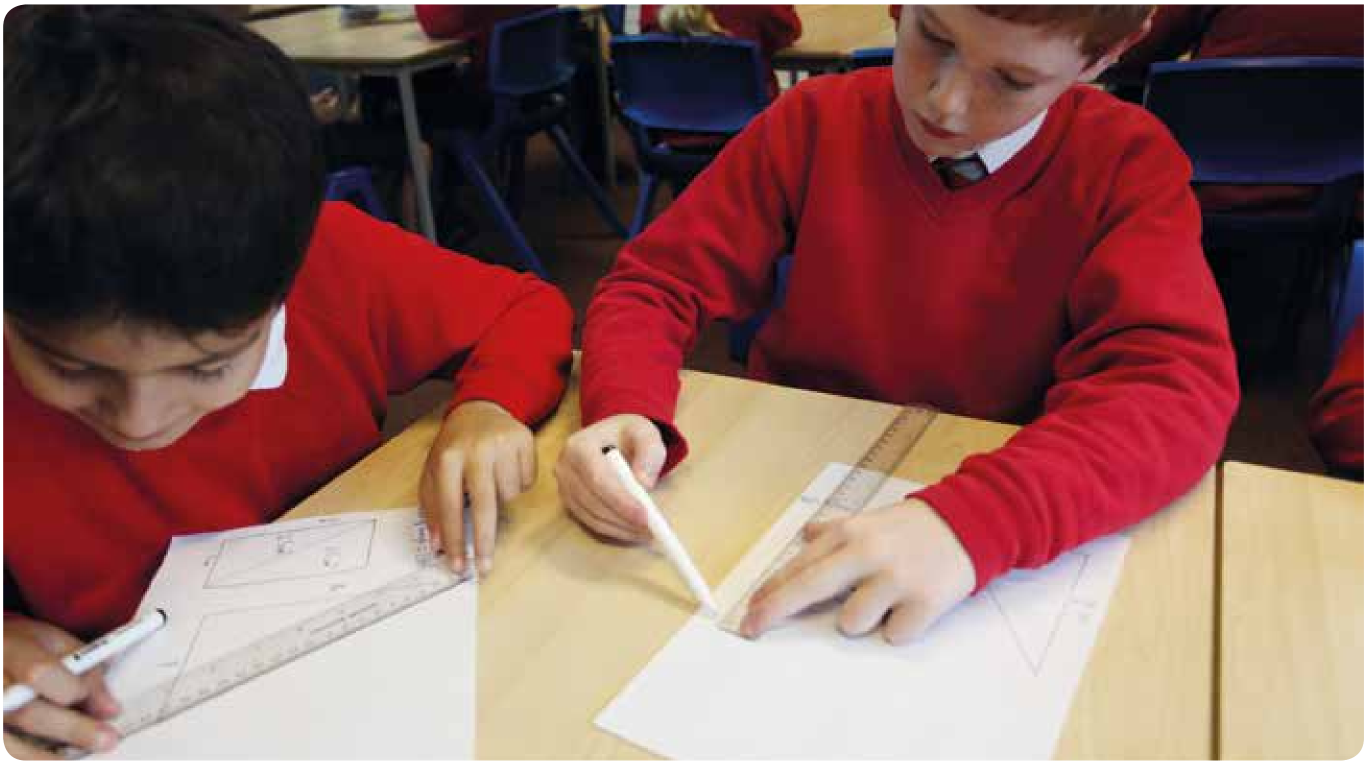
Finally, once scales that have all the previous properties *and* a fixed 'zero' boundary are introduced, *ratios* of values become meaningful comparisons. This relates to children progressing from 'additive' thinking to the crucial further possibilities of 'multiplicative' thinking in relation to measures.

This shows that there is a close correlation between Stevens' proposed classification and the development of measuring and statistics with children. First, we invite children simply to **discriminate** between various qualities (general sorting;

12 This is actually because in practice there is no necessary, fixed 'zero' point on an interval scale, and both positive and negative values can potentially be thought of as 'going on forever' either side of an arbitrary 'zero'. In later physics, children may meet the Kelvin temperature scale with its 'absolute zero' and also the idea of a 'Big Bang' and 'zero time'.

13 Interestingly, it is the importance of 'starting from zero', e.g. when we measure length with a ruler, that children often do not appreciate immediately.

14 We will also leave discussion of negative values within quantum physics and the notion of 'anti-matter' for another occasion; we're just doing primary mathematics at the moment!



noting qualities such as heaviness, extension, heat and so on). Then we introduce the idea of **ordering** various degrees or levels of those qualities (**comparing**: e.g. longer than, shorter than and so on). Next, we introduce *equal* ordered degrees of difference in qualities (that is, intervals or **units**), and thus the possibility of adding and subtracting (comparing) differences and *naming* individual amounts (**values** on a scale). Finally, with ratio scales we introduce the full possibilities of comparing (adding and subtracting) values themselves and **ratios** of values with multiplicative thinking.

Together these activities and objects (in bold, above) constitute the key aspects of measuring in the primary years, and can be used as key focuses for attention both in teaching and in judging children's progress. These are the *roots* of measuring activity. Notice how in teaching, these key focuses are approached in the same order as Stevens' cumulative classification of scales.

A special note about money

Stevens' classification has been valuable for the amount of productive debate it has created, but a weakness is possibly revealed when we ask which type of scale applies to money. In terms of *cash*, money is measured with a ratio scale; you can't have 'negative cash'. But of course you can *owe* money, and that makes money seem more like an interval scale; 'zero' money becomes a fairly arbitrary point when you're a student, and both debt and riches appear to stretch potentially endlessly in either direction. Some *ratio* comparisons still work. £10 is still 'twice as much as' £5, and owing £10 means owing 'twice as much as' owing £5, but 'how many times' as much money have you got if you *have* £10 as opposed to when you *owe* £10?

It is worth noting as well that physical money is not continuous in the same sense that time, length, area, volume, mass and so on, are considered to be. *Actual* amounts of money that we handle are exact, whereas money used in financial calculations such as currency conversions is often treated as continuous, e.g. £1 = \$1·151715…

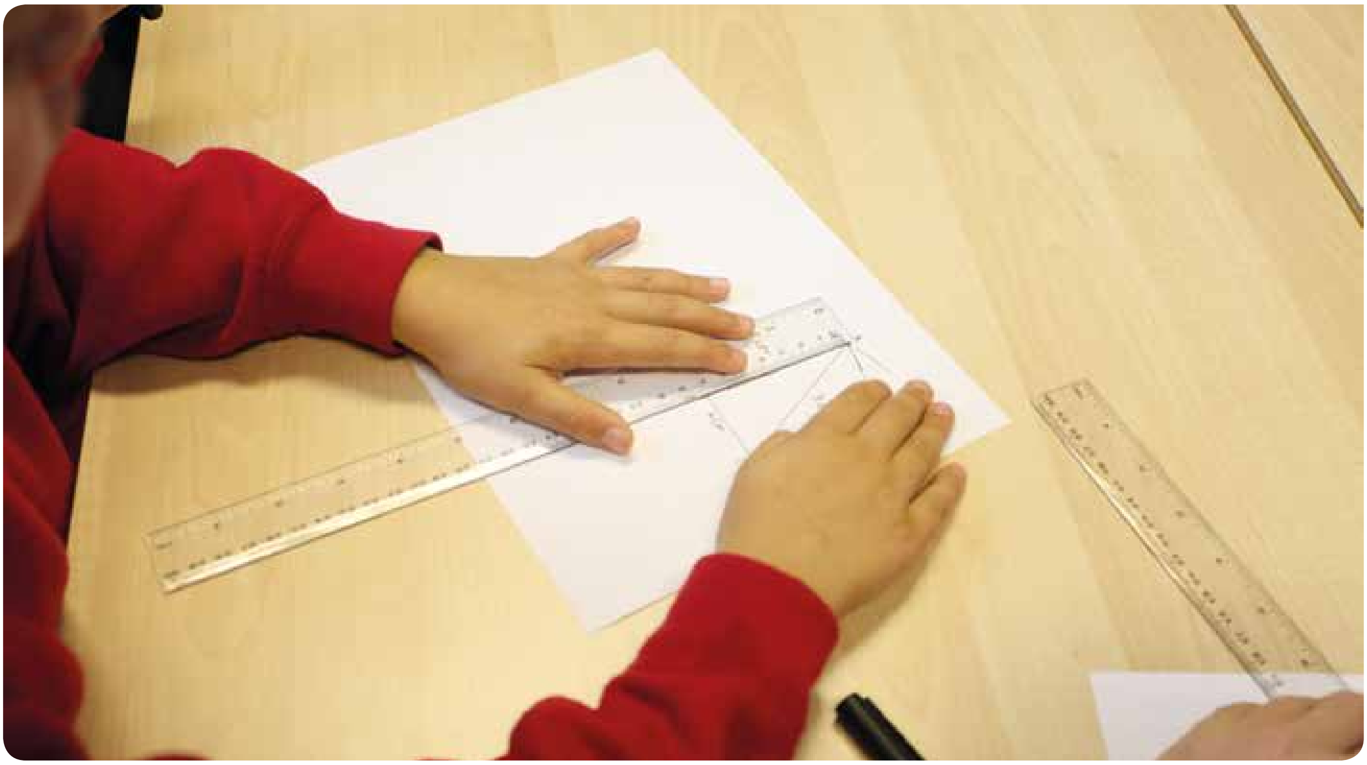
Money is included as a 'measure' (of economic worth) in the curriculum, but the ways in which it is used and calculated are significantly different to other, physical measures.

The physical measures

The physical measures introduced during primary schooling in England are: length, mass and weight, volume and capacity, time, speed, temperature and area.

Length and distance: Technically, when we measure 'length' we measure what would perhaps be better called 'linear extension'. Confusingly for children, in everyday life linear extension gets called different things in different contexts. Height, width, depth, length and distance are all different ways of referring to the same quality of linear extension, and so children need to connect references to their 'height' and how 'tall' they are with the 'depth' of a swimming pool, the 'width' of their bedroom, the 'length' of a football pitch, and with how 'far' it is to the shops, as all measures of 'the same thing'. Children will need much discussion time about this great variety of language use, and also around the wide variety of instruments used to measure different 'lengths' and 'distances' in different contexts.

Gradually, children will learn that there is also an important distinction between 'distance' and 'displacement' when measuring 'how far' it is from A to B. 'Distance' is simply an



amount or a magnitude, e.g. how far you actually have to travel, whereas 'displacement' is both a magnitude *and* a direction. This is called a vector generally, and a 'translation' in geometry. In everyday life we describe the displacement between two places as the linear distance between them 'as the crow flies'. We assume crows fly along the shortest (straight) path between two points, whereas in reality, e.g. the distance from our home to school will be further than 'the crow flies' because we won't be able to travel in a straight line. Because displacement is a straight-line path, we are able to specify it as movement in a constant direction. This distinction is obviously crucial in answering, 'How far is it from A to B?'.

There are interesting later developments in measuring 'distance' on a global scale; the shortest distance between, say, London and Los Angeles lies along what's called a 'great circle'.[15] This is a section, or arc of a circle drawn around the Earth with its centre at the centre of the Earth. Aircraft generally navigate along great circles, but typically if we are asked to say how far away Los Angeles is from London we are more likely to say it's '11 hours' away, than 5,452 miles. This also connects with the measuring of astronomical distances in 'light years'; when distances are large, the 'distance' from A to B becomes more meaningfully expressed in lengths of time than in units of 'linear extension'.

The standard (SI) unit of linear extension in all contexts is the metre (m).[16] Length is measured with ratio scales (metric or imperial), since 'zero length' is an absolute. Consequently, ratios of lengths to each other make good sense and are used frequently in both everyday life and in science.

Mass and weight: The 'mass' of a physical object is the pleasingly simple idea of 'how much of it' (that is, the material stuff) you've got; measuring this directly however, is not so simple. In practice, as Isaac Newton (1642–1727) pointed out, we take advantage of the fact that under the effect of gravity, *weight* and *mass* are directly proportional to each other. If you double the 'amount of stuff' you've got you will find that it now weighs twice as much as it did.

What this means is that (unless we are out in space) we can compare the masses of two objects with each other by comparing their weights. If a proud new father weighs 22 times as much as his newly born daughter, then we can be sure that his 'mass' is 22 times greater than hers (there is 22 times 'as much' of him as there is of her).

The standard SI unit of mass is the kilogram (kg). In order to measure mass we do in practice compare objects with this standard unit by 'weighing' them – that is, by comparing their weights with the *weight* of a 1 kg mass. 'Weight' however is a force; it is the gravitational force acting upon any object. In imperial units it is measured in 'pounds' (lb) and 'ounces' (oz). In the metric system, the force due to gravity, that is, 'weight', is measured in 'newtons' (N).

The designers of space stations have to do their calculations based on the knowledge that, in orbit, the masses of everyone involved will not have changed, even though their weights will have. The force of a collision between two astronauts in space does not change however (it hurts just as much), because their masses do not change. Even though

15 Or it would be if the Earth were actually a sphere; for all practical purposes we treat it as if it is.

16 The International System of Units (known as SI) is a standard system of units of measurement, used by the international scientific community.

each is weightless, two astronauts' bodies still have as much momentum when moving in space as they did on Earth (and therefore will take just as much stopping).

In school, using the correct language for the metric system can sound odd because it is not the everyday language children meet outside school. In the everyday world, we do compare masses by 'weighing' them, but technically we should not go on to say that the 'weight' of something is 'so many kilograms' – that's its *mass*.

So we teachers have something of a problem in deciding whether to use the scientific language of physics as we talk about the SI units of mass (kg), or whether to carry on talking about the 'weights' of objects in kilograms and – in everyday language – pretend that kilograms are units of weight. Many teachers talk of 'heaviness' to avoid using the word 'weight', and do indeed ask children to 'compare masses'. We recommend that you follow common everyday language use until children address 'mass' and 'weight' in their science lessons.

Both mass and weight are measured with ratio scales, since their 'zeros' are absolute. Ratios of masses and weights to each other make good sense and are used frequently in both everyday life and in science.

Volume and capacity: 'Volume' is the amount of space something occupies, whereas 'capacity' is how much space there is inside a vessel or container of some sort, or how much volume it could 'hold'. In the metric system, the volumes of liquids are usually measured in litres (e.g. drinks, petrol), and the volumes of solid objects in cubic metres (m^3). Capacities are typically measured in m^3, but can also be expressed in litres (e.g. a 1 ℓ bottle).

1 litre (ℓ) is equivalent to 0·001 m^3 (or, one thousandth of a cubic metre). Measuring either volume or capacity in m^3 introduces children to what is called a 'derived' measure; the unit of volume (or capacity) is *derived* from the so-called 'base measure' for length (m).[17]

Interestingly, 1 ℓ of pure water has a mass of 1 kg. Both 'volume' and 'mass' are in a sense measures of 'how much' of something you've got. 'Mass' is the basic scientific SI unit for how much 'matter' there is, whereas 'volume' judges 'how much you've got' by measuring the amount of space something takes up. In everyday life we tend to use 'weights' and 'volumes' for all practical purposes, e.g. buying and selling, cooking and so on.

In science, where 'how much matter' (or material substance) you are dealing with is generally much more important than the space it is occupying, we use kilograms to measure how much 'material' there is of an object. When we want a drink, we are generally much more interested in how much volume we will be given.

The capacities of containers and vessels are usually measured by calculating using the three dimensions of space (length, width, and depth), or by filling them with a known volume of liquid. Capacities are then typically quoted in both m^3 and in litres.

The volumes of liquids are usually measured inside graduated containers (or through pumps), or in the case of

17 For this reason it is usually more effective to introduce children to litres before 'cubic' measures for volume and capacity. Calculating with *derived* measures (e.g. m^3) depends upon a good skill with multiplying. Otherwise the effort of the arithmetic involved may prove a distraction for children.

solid objects of awkward dimensions, volumes of objects can be measured by using a displacement vessel.

Both volume and capacity are measured with ratio scales, since their 'zeros' are absolute. Consequently, ratios of volumes (and capacities) to each other make good sense, and are used frequently in both everyday life and in science – particularly in the repetition of mixtures in specified proportions.

Time: There are several aspects to the topic of 'time': telling the time, duration, succession and speed. As a compound measure, '**speed**' is addressed formally in the final year of primary schooling, although it will certainly come up in children's everyday conversations from an early age as they are asked to 'go faster' or more 'slowly' or 'quickly', and as they experience 'racing' in a variety of forms.

There are two central ideas that children need to connect together for effective development of their understanding of how we measure time: the idea of *linear* time and the idea of a repeating *cycle*. Linear time is the sense that 'time' stretches both backwards into history and pre-history forever, and endlessly forwards into an infinite future. The key visual illustration of this is of course the 'timeline' – which bears an uncanny resemblance to the number line, even down to the fact that there is an identified 'zero' point for most cultures, 'before' and 'after' which 'things were different'. In the activity groups we refer to these times as BCE and CE: before common era and common era.

Work with timelines connects well with children's developing understanding of numbers as distances along a number line, and with their appreciation of different kinds of numbers. The ideas of 'order' and 'succession' are crucial in this.

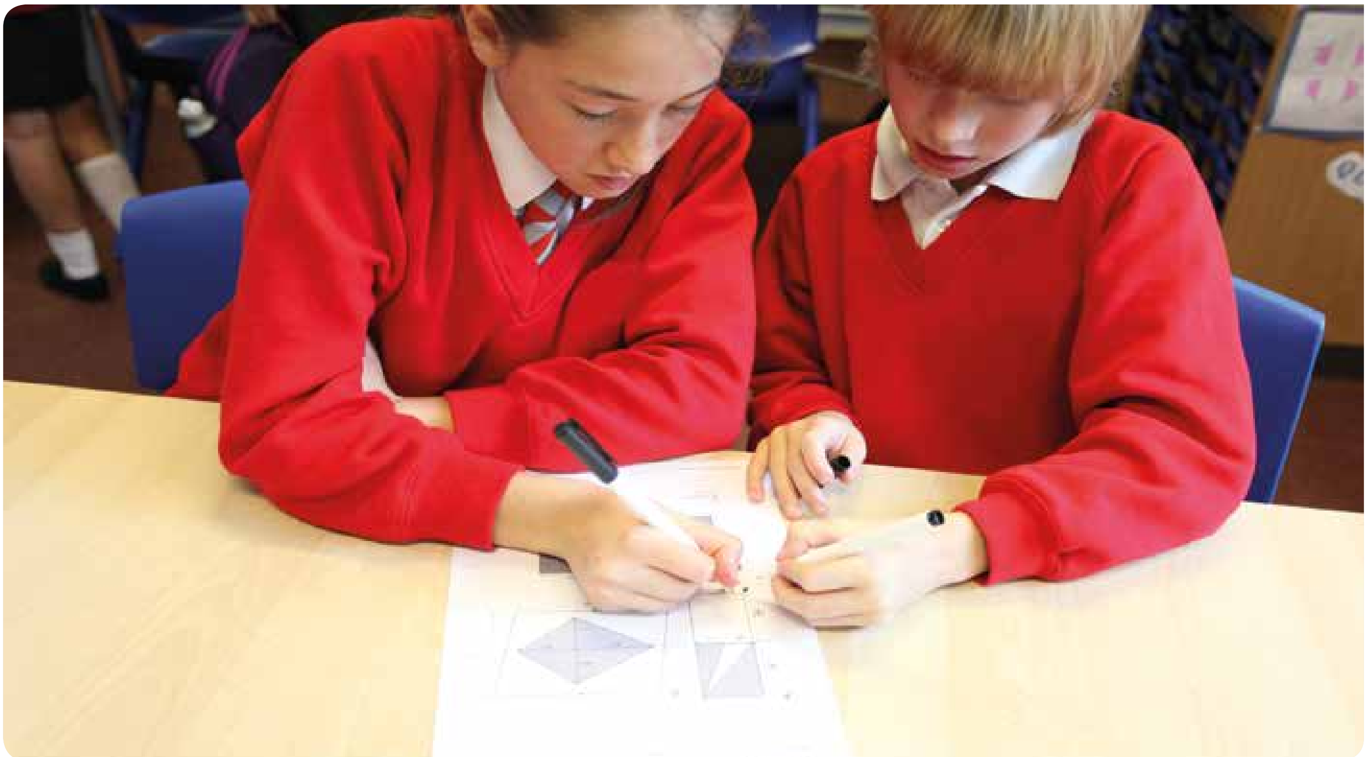
Since any timeline includes an 'arbitrary' zero point (it could have been chosen differently), the line becomes an *interval* scale stretching potentially infinitely either side of zero. As such, ratios of dates to each other don't make sense, only ratios of *lengths* of time (durations) to each other. 4 hours is twice as long as 2 hours, but 4 July is not 'twice as' anything in relation to 2 July.

The idea of repeating 'cycles' is closely related to our experiences in a constantly changing natural world of moon cycles, daily (sun) cycles and seasons. We use these natural cycles in our everyday measuring of time as we plan 'summer' holidays, prepare for important dates with our calendar, sleep 'at night' and so on. The hands of an analogue clock illustrate to children constantly how we measure time in repeating cycles.

Since both linear time and repeating cycles are crucial to how we deal with succession, duration and telling the time, children need *both* kinds of illustration prominently as we work with them on measuring time.

Because the scale of time ranges so widely from instants to eons, and because our awareness of how fast 'time is passing' changes so much depending on context and mood, we sense time more indirectly. This means that children are also particularly dependent on discussion, illustrations and active experiences with instruments to develop their personal understanding of time.

The standard (SI) unit of time is the 'second' (s).



Temperature: Temperature is a measure of how 'hot' or 'cold' something (or someone) is, and is measured in degrees on the Celsius (°C), Fahrenheit (°F) or Kelvin (K) scales.[18] In Europe and in much of the world the Celsius scale is generally used in everyday life; in the USA the Fahrenheit scale is commonly preferred. In much of science, the SI unit of K (Kelvin) is used. Degrees Kelvin are equivalent in size to degrees Celsius, but the zero of the Kelvin scale is a theoretical 'absolute' zero meaning there are no 'minus' temperature values possible in K.

The Celsius scale is based around the freezing point (0 °C) of pure water at one atmosphere of pressure, or sea level, and the corresponding boiling point (100 °C). Thus children will probably first experience the idea of 'negative' values (or 'minus' numbers) in the context of 'below zero' winter temperatures in either °C or °F, and/or ice in everyday life, and this is a very helpful context to use when introducing negative numbers.

Since the Celsius scale includes an 'arbitrary' zero point (that is, it could have been chosen differently), it is an *interval* scale stretching potentially infinitely either side of zero.[19] As such, ratios of individual temperature values in °C or °F to each other don't make sense, only ratios of temperature *differences* on these scales. A drop of 6 °C is twice as big a drop as one of 3 °C, but ⁻6 °C is not 'twice as cold' as ⁻3 °C.

Children are very well aware of temperature through their senses from an early age, and so work in primary school is mainly directed to introducing the Celsius scale and experiencing various types of thermometer. Many children will be familiar with mercury or alcohol-in-glass, digital and liquid crystal thermometers simply from their own experiences of illness, but it is helpful for them to meet and discuss these different instruments in school as well.

Area: Area is usually the second 'derived' measure that children meet, after volume. The SI unit for area is the 'square metre' (m^2), derived from the base unit for length (m).

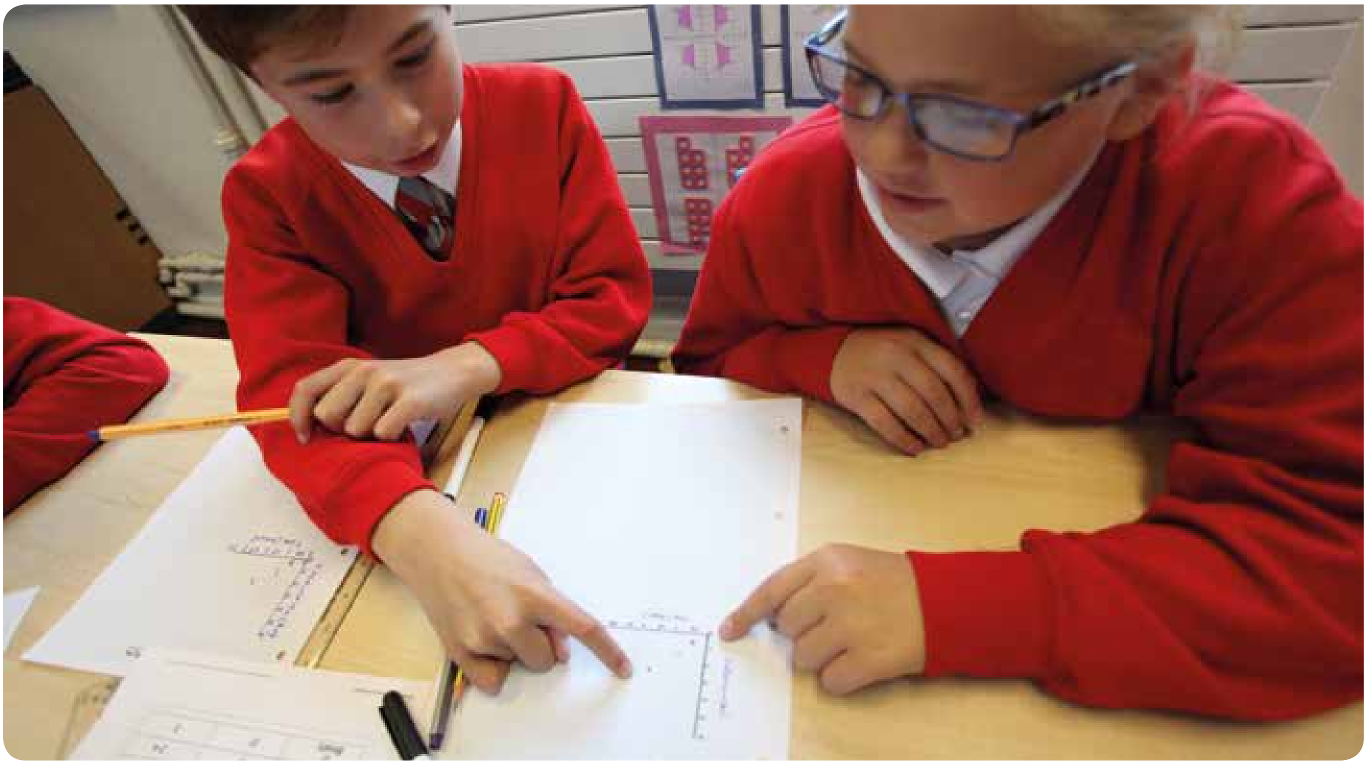
The intuitive idea underlying our conception of area is that of 'surface', and children need plenty of active experience with surfaces before attempting to measure them. Covering surfaces is a helpful activity, so painting, tiling, jigsaw puzzles and so on, are all beneficial early experiences.

Since area is measured with a derived unit there are strong connections between understanding multiplication as an arithmetic operation and work on measuring area. The two dimensions of any flat surface correspond helpfully to the two numbers involved in the binary operation of multiplying. Visual areas are a most effective illustration for many properties of multiplication and multiplying is essential to the measuring of area.

Since there is such a close connection between multiplying and measuring area, formal work on measuring area is usually timed to coincide with a suitable stage in the development of children's calculating skills.

18 Technically, it is a measure of the thermal energy per particle of matter or radiation; it seems the 'hotter' something gets, the more those little things get agitated.

19 This means that as far as the Celsius scale is concerned, temperatures could go on getting ever hotter or colder 'forever'. Only the Kelvin scale acknowledges explicitly that in our universe it is presently thought impossible for temperatures to become any colder than ⁻273·15 °C.



There are many interesting facets of our natural world to explore that relate surface areas with volumes and the significance of their relationships to biological life, e.g. babies and small creatures have *proportionally* much more surface area to their bodies in relation to their volume than adults and larger creatures. This is one key reason why babies and small creatures are much more vulnerable to extremes of heat and cold than adults and larger creatures.

Measuring in Geometry, Measurement and Statistics 6

There is much reinforcement and use of previously introduced measuring ideas at this stage, in particular conversions between units and the use of measuring instruments for the actual measuring of angles, lengths and areas as both 2D and 3D shapes are constructed. There is also however, a notably increased emphasis in Geometry, Measurement and Statistics 6 on the use of formulae to calculate areas, lengths and volumes, on angle sums to calculate angles, and on the use of scale factors for calculations about enlarging and reducing lengths, areas and volumes.

At this stage as well, there is increased use of established measures for calculating within problem solving situations. For example, work on speed not only introduces a *compound* measure relating time and distance (that is, 'speed' itself), but this is also achieved within a context of average speeds and an introduction to the statistical 'mean' (Measurement 1). Mass also features within a context of average product quantities (Measurement 1).

As an extension to the investigating of scaling transformations, children are invited to explore what happens to lengths, areas and volumes as both 2D and 3D shapes are scaled up or down (Measurement 4). They learn that these magnitudes change at different rates when scaled, for example if you scale the lengths of a shape by a particular scale factor, then its area will multiply up by the *square* of that factor.

As in earlier work, wherever possible measuring activities are set within realistic contexts so that children realize that measuring in everyday life is always done for a specific purpose and that in life we always work to a level of accuracy that suits the purpose of the particular occasion.

Length and distance: The two key aspects of length and distance worked on in Geometry, Measurement and Statistics 6 are the handling of translations with Cartesian coordinates, in which children need to remember that the distance involved in a translation is directly (as the crow flies) between the beginning and end points, and children's first investigation of *curved* distances in the circumference of a circle. Both of these aspects set children up for the idea that distances between two points may be of many kinds (Geometry 3).

Children continue to be expected to work with, and convert between, metric units of many kinds, and distance crops up again linked with time in measures of speed (Measurement 1).

Mass and weight: Children work on mass and weight in the context of the '℮' (for 'estimated') symbol on food packaging, and the regulations relating to labelling and average quantities of foodstuffs, e.g. soup (Measurement 1).

Volume and capacity: In Geometry, Measurement and Statistics 6, children build upon the introduction in Geometry, Measurement and Statistics 5 of volume as a 'derived' measure, that is, a calculated amount of 3D space in (e.g.) cm^3 as opposed to 'liquid volume' measured in ml, and begin to use the formula $V = l \times w \times h$ for calculating the volumes of cuboids (Measurement 4).

Work on liquid volumes is continued as children consider 'average' volumes of liquids in the context of commercial packaging.

Time: Time is addressed within work on the compound measure 'speed' in Geometry, Measurement and Statistics 6. The continuous nature of time is further explored in discussion of continuous distance/time line graphs that involve time as one variable. Conversions between units of time occur within the context of comparing different measures of speed, e.g. converting between mph and km/h (Measurement 1).

Area: The measuring of area is further developed from the work of Geometry, Measurement and Statistics 5 through exploration of the areas of a wider range of 2D shapes in Geometry, Measurement and Statistics 6. Children continue to be invited to divide up 2D rectilinear shapes into collections of smaller shapes whose individual areas may be calculated and then totalled.

Importantly, after an exercise with tangrams in order to emphasize the conservation of area, in Geometry, Measurement and Statistics 6 children are introduced to formulae for calculating the areas of triangles and parallelograms. These are then put to use in order to calculate the areas of more complicated and irregular shapes (Measurement 2).

Angle: In the context of constructing 2D shapes, children further explore angle sums of shapes, including triangles and parallelograms, and are introduced to 'vertically opposite' angles and their equality. Using vertically opposite angles allows children to calculate missing angles within a wider range of contexts (Geometry 1).

Children also explore important relationships between angles and lengths of related sides, for example the longest side of a triangle will be opposite the largest angle.

Speed: Speed is most probably the first compound (or sometimes 'composite') measure that children will study formally, although of course by the age of 10 or 11 years children will have already studied and experienced speed directly, personally and informally for many years.

Essentially, a compound measure is one that relates two other measures of different types together; in the case of speed, we relate distance and time to each other, and measure it in, for example, miles per hour or metres per second. Other compound measures include density (mass divided by volume, e.g. grams per cubic centimetre) and pressure (force divided by area, e.g. pounds per square inch).

Importantly, because there are these two aspects, or 'dimensions', to measuring speed we can illustrate speed graphically with the two dimensions (axes) of a Cartesian graph. So conventionally we can plot changes in distance (y-axis) against changes in time (x-axis) on a graph, and the gradient of the line at any point, or over any period, will represent the rate at which something was travelling, or its 'speed' (that is, ratio of distance to time, or distance ÷ time). This has the visual advantage of showing a faster speed as a steeper line on a graph.

Arithmetic mean (or 'average'): The arithmetic mean (or average) of a set of data is one of three so-called 'measures of central tendency'. These are attempts to give a single 'summary' value for a group of data that will allow groups of data to be 'summed up' in a single value, and then in some contexts compared with other groups. The other two measures of central tendency are the median and the mode.

The arithmetic mean is calculated by dividing the total of all data values in a given set by the number of such values in the set.

If we wanted to know whether children were gradually growing taller in each generation we would need to compare the 'average' (or mean) heights of all the children of each successive generation, thus summarising the individual heights of each generation in order to compare whole generations with each other.

Technically, because the mean is calculated using all the data values of a given set it will be affected by any extreme values (or outliers) judged 'untypical' that will 'skew' the summarising of the data set unreasonably.

Statistics in the primary years

As with measuring, no one ever uses statistics in real life without a specific *purpose* to it. In practice there is always some question or questions that people are trying to answer. Thus in work with children, any work on statistics should always be led by a key question (or questions) that shapes and directs the work itself.

The simplest of beginnings to statistics can be made with collections of real objects (e.g. things children bring back from a nature trail walk), about which a teacher could ask, 'So, what have we got here, then?' Children can be encouraged to sift through, discriminate between and identify various categories into which objects can be sorted. Answers then become something like, 'We've got some of these, a few of those, only one of those' and so on. The objects themselves can then be displayed, grouped within their categories and suitably labelled.

Note how closely this activity relates to the use of 'nominal' scales in measuring. There is no ordering of objects or categories involved, simply a *discriminating between* objects. Such work is the beginning of handling **categorical** data.

Thus sorting through a collection of real objects, noting similarities and differences and organizing the objects into categories in order to respond to a particular question, comes before similar activities with collections of data. The important thing is that the statistical activity is purposefully directed towards answering an initial specific question.

The data-handling cycle[20]

Purposeful statistics begins with a specific question (A) being posed. At the next stage (B) we identify the kind(s) of data that would help answer the question. Then the chosen data is collected (C), and organized (including any visual representation) so as to reveal any patterns of significance relevant to answering the question. At stage (D) we interpret any patterns found at (C) in order to return and try to answer the question at (A).

It is always best to choose an initial question within an area of topical interest to the children. So, for instance, during

20 See Graham, A. (1990) *Supporting Primary Mathematics: Handling Data*, Milton Keynes: The Open University, for a similar version.

some work on road safety children might be being advised to 'walk straight across' roads, rather than to cross them diagonally. Discussion could raise the point that it takes longer to cross a road diagonally than it does to go straight across, and therefore crossing diagonally puts a pedestrian in potential danger for longer. A relevant question (A) could be, 'How much more dangerous is it to cross a road diagonally than it is to walk straight across?' Children could then (B) discuss what kinds of data might help them to answer the question, and come up with the idea of marking out a 'road' on the playground and timing each child in the class both walking straight across and walking diagonally. Collecting and organizing the data so that comparisons can be made (perhaps involving the averaging of times) follows at stage C, and then discussion (D) could interpret whether the time differences found at C help to answer question A well enough.

It may be that children feel the first run-through of the cycle answers the original question satisfactorily, in which case the work is over. Quite often, however, a first run-through reveals that the initial question was not quite 'fit for purpose'. The question is then refined, and the whole cycle run through again on the basis of a different question at A. It may be that in the road safety example, children decide it actually matters more *where* you cross a road, than whether you go exactly straight across. In this case a different question can be raised at point A, perhaps timing how much warning you have of impending traffic if you try to cross a road near a bend.

In this way we gradually understand the purpose of the statistical activity by increasing refinement of the task as we undertake repeated cycles. Children get to appreciate how important it is to ask the right question in the first place, as well as a range of data-collection techniques, data-organizing techniques, forms of pictorial and graphical representation, and approaches to interpretation, as different cycles are pursued.

In this important way, children come to realize that data handling and statistics are always undertaken for a specific purpose. That is, we do not construct graphs from data and then decide what questions those graphs could answer. We begin with a question, collect data, and then construct a graph to show that data can help us to find the answer to the question.

Connections with measuring

Early measuring and early work on statistics are virtually indistinguishable from each other. In both types of activity children are learning to *distinguish between*, to *discriminate*, and to identify qualities that may be measured and related. Of course how various qualities may be related to each other will eventually take children to algebra and the study of relationships *in general*, and any calculating involved in statistics automatically relates back to their work in number. Everything is connected, in doing mathematics.

As children collect and analyse different kinds of data, in different kinds of ways, to answer a variety of questions, they will learn to *distinguish between* qualities, between *ordered* and un-ordered data, they will need *units* and utilize interval scales, and *compare* amounts multiplicatively using *ratio* scales. In all these ways the development of children's measuring is woven within the development of their work in statistics.

Statistics is a key context within which measuring demonstrates its own clear purposes.

Statistics in Geometry, Measurement and Statistics 6

There are three main areas of statistical study in Geometry, Measurement and Statistics 6: linear graphs relating two continuous variables (including a linear graph relating miles to kilometres), the introduction of a first 'measure of central tendency' (the arithmetic mean), and further work using pie charts, utilizing children's ongoing work with angles, fractions and percentages (Measurement 1 and Geometry 1).

The data handling activities are all set in contexts that lead with an initial question, giving the work purpose and therefore the key criterion for success: does the data handling undertaken in each context actually offer a useful answer to the original question?

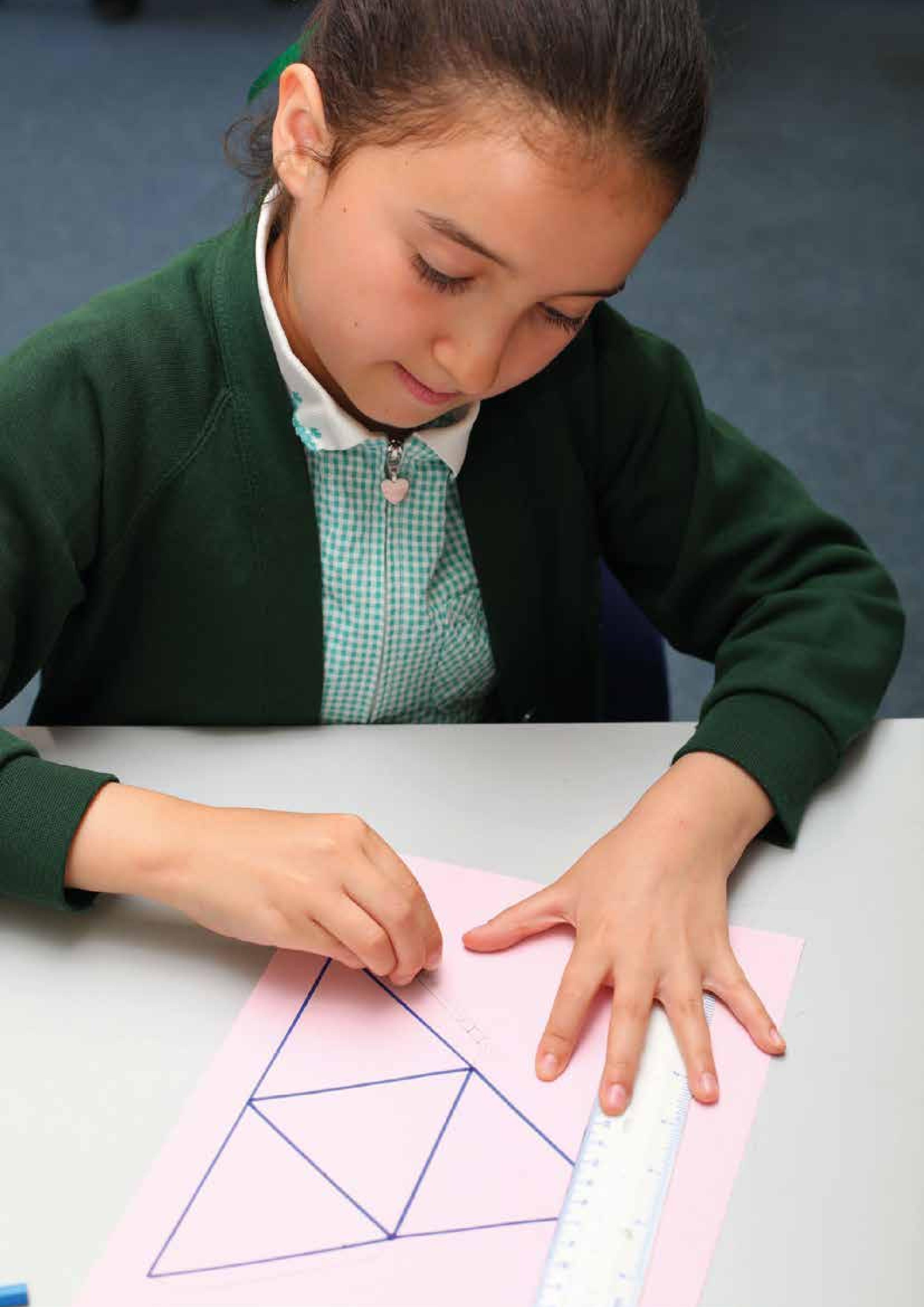
As in Geometry, Measurement and Statistics 5, there are important connections between this current Geometry, Measurement and Statistics 6 work in data handling and ongoing work in measuring, geometry, and in number. Work on coordinates, axes and scaling in geometry and measures ties in with work on continuous line graphs, and the interpretation of points on continuous line graphs involves important use of number work on fractions and decimals. Interpreting a continuous line graph also involves the important experience of acknowledging *approximate* answers, which ties in well with all work on measuring.

The idea of an 'average', in particular the arithmetic mean, is introduced within the everyday context of food packaging in which actual amounts of a foodstuff may vary from packet to packet but labelling has to guarantee variation within strictly defined limits. This introduces explanation of the regulatory '℮' symbol common on many commercial labels.

Subsequently, the idea of average is further developed in the context of calculating 'average speeds'. This links the idea of an average with work on linear graphs.

The context of speed is also utilised in covering work on conversions between systems of measures in Geometry, Measurement and Statistics 6, relating measures expressed in km/h to mph – with which children will typically be much more familiar.

Continued work on pie charts at this stage relates well to the continuing work on angles (measured in degrees) in geometry, and also to work on percentages, ratio and proportion in number. Pie charts are again used within a context in which it is clear that *proportions* within a whole are being stressed.



Glossary

Most mathematical terms used in this *Geometry, Measurement and Statistics 6 Implementation Guide* and *Geometry, Measurement and Statistics 6 Teaching Resource Handbook* can be found in a good mathematics dictionary such as the *Oxford Primary Illustrated Maths Dictionary*.

Other terms you might not be familiar with or which may provide particular challenges for children are explained in this glossary.

angle

The amount of turn between two straight lines that have a common end point (vertex) or intersection.

bar chart

A chart in which data is presented as equal-width rectangular ‘bars’, each representing a different category, with the length of the bar indicating the value (see Fig. 1). The bars can be vertical or horizontal. The length of a bar is not restricted to fixed intervals.

base-ten apparatus

A set of concrete materials, systematically designed to help children understand our place value system. Small cubes, sticks of ten cubes, flat squares of 100 cubes and large cubes of 1000 small cubes are used when talking about ones, tens, hundreds and thousands respectively (see Fig. 2).

bridging

Bridging is a calculating technique that involves partitioning (splitting) the number to be added or subtracted. For example, bridging across 10 exploits the adding and subtracting facts to 10 that children learn early on, e.g. $8 + 9 = (8 + 2) + 7 = 17$. Bridging can be used across any number, and in the context of measurement might involve bridging across the nearest whole unit, e.g. 1 kg when working with masses expressed in kilograms, or kilograms and grams, or the nearest hour when adding or subtracting a given time interval to find a new time.

Bruner

Jerome Bruner (1915–2016) was an extremely influential and distinguished psychologist. His distinctions between enactive, iconic and symbolic representation have been particularly influential in the field of education.

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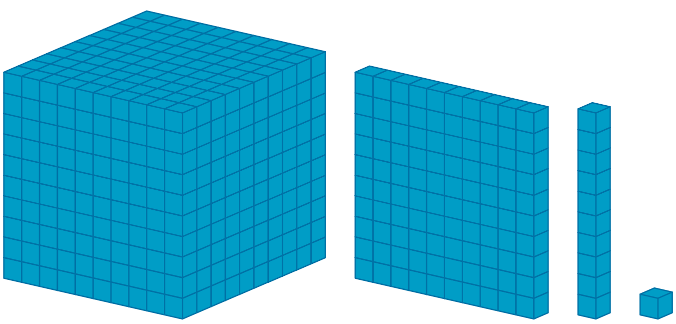

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communication mediator

A communication mediator is an object, image or action that helps communication in some way. These communication mediators need to be carefully introduced to children, e.g. Numicon Shapes and number rods become communication mediators when they help to illustrate discussions about the relationships between numbers, coins and lengths of time. However, there’s no ‘magic in the plastic’ – a physical object or image is just a physical object or image unless it is actually supporting communication.

diagonal

A straight line joining two vertices of a polygon (or polyhedron) that is not a side (or an edge).

edge

Used to refer to a ridge on a 3D shape at which two faces or surfaces meet. A cube has 12 edges, as shown in Fig. 3. Initially children may talk about both straight and curved edges; later, ‘edge’ refers only to boundaries of shapes that are straight lines. (See also **side**.)

enactive, iconic and symbolic representation

Jerome Bruner (1966) distinguished three key ways in which we humans represent experience to ourselves: through enactive (internalized action), iconic (sensory) and symbolic (e.g. language-based) representations. In Numicon we seek to combine all three forms of representation so that children experience mathematical ideas through action, imagery and conversation.

face

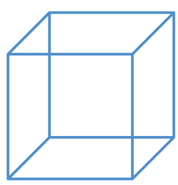
The flat **surface** of a 3D shape. A cube (see Fig. 4) has six faces.

flat 2D shape

A two-dimensional (2D) shape, such as a rectangle, hexagon or oval. ‘Flat’ provides a physical and visual description which helps children distinguish or sort shapes of this type, and hence to develop mental imagery and generalize about them. (See also **solid 3D shape**.)

generalization

A statement or observation (not necessarily correct) about a whole class of objects, situations or phenomena. Generalizations are essential and everywhere in mathematics – numbers, for example, are generalizations, as are shape names. For this reason children need to generalize and to work with generalizations constantly.



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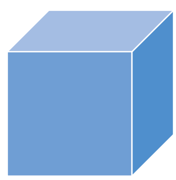

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length

Length may refer to a dimension of an object, and in this sense be used to distinguish from width or breadth, height or depth, thickness, or distance. Alternatively, and perhaps confusingly for children, it can be used as a general term to describe all these measures of dimension.

line graph

A graph that presents data as a series of points joined by lines, often showing change in a measurement over time, as in Fig. 5.

mass

The amount of 'matter' in an object, which gives it heaviness or **weight** under gravity. The mass of an object is found by weighing it.

non-standard unit

In Geometry, Measurement and Statistics 1, children chose units to explore length, mass and capacity, e.g. a distance in steps, or a capacity in scoops. These are non-standard units both in the sense that their size is not widely agreed (and may not even be fixed), and in the sense that they are not commonly used for making exact measurements. In subsequent Teaching Resource Handbooks, children work towards understanding and using 'standard' units of measurement such as millimetres, kilograms or litres.

number fact

The term 'number fact' usually refers to an operation on two or more numbers, together with its outcome. So '6 + 3 = 9' is a 'number fact', as is '256 ÷ 16 = 16'. In UK schools simple addition and subtraction facts are often referred to as 'number bonds'.

number names/objects/words

Adults often use number words such as 'four' or 'twenty-three' as nouns, and ask children questions such as 'What is seven and three?' In our language, nouns name objects, so we commonly (and unconsciously) assume that number words must be being used to name number objects – thus numbers are often treated as if they are objects.

It is important to remember that we don't always use number words as nouns; quite often we use those same words as adjectives, as in 'Can you get me three spoons?' One of the key puzzles for children to solve is when to use number words as adjectives, and when as nouns.

number sentence

The metaphor of a 'sentence' – in the sense of a unit of language – is sometimes used to describe a number fact written horizontally, left to right, as in '4 + 23 = 27'.

numerals

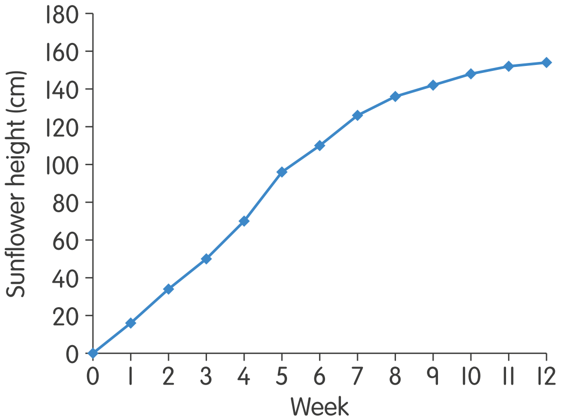
Numerals function as shorthand for number words. The numeral '5' is shorthand for the word 'five'. When used as a noun, the numeral '5' is often said to represent or stand for the number object we call 'five'.

Numicon Shape pattern

The system of arranging objects or images (up to ten in number) in pairs alongside each other; this is also sometimes called 'the pair-wise tens frame'. Fig. 6 shows the Numicon 7-pattern.

Numicon Shape

Numicon Shapes are pieces of coloured plastic with from 1 to 10 holes, arranged in the pattern of a pair-wise tens frame (see Fig. 7).



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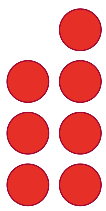

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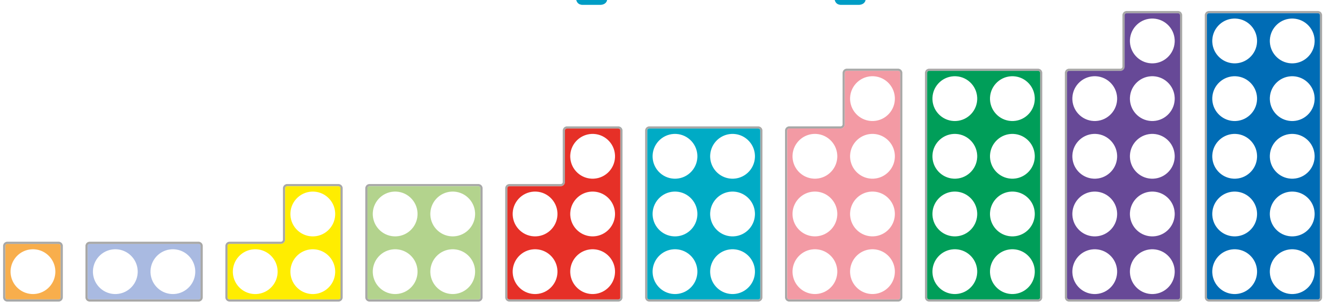

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oblong

A **rectangle** which is not a square – that is, a 2D shape with four straight sides and four right angles in which one pair of sides are longer that the other pair (see Fig. 8).

parts and properties

The **properties** of a shape are defined by how its **parts**, e.g 'vertices' and 'edges', relate to each other. For example, one property of a trapezium is that it has just two sides that are parallel to each other. A property of a rectangle is adjacent sides meeting at 90°.

Piaget

Jean Piaget (1896–1980) was a philosopher and psychologist who spent years studying young children and how they learn. He believed that the development of knowledge is based on developing cognitive structures and that children should be actively involved in their own learning.

pie chart

A chart in which data is shown as proportions of a circle. The slices of the pie chart represent the proportions of each item within a whole data set (see Fig. 9).

prism

A 3D shape with faces which are all polygons and with a cross-section which is the same along its length. The cross-sectional shape is often used to name the prism, e.g. Fig. 10 shows a triangular prism. Note that a cylinder, although very like a prism, is not strictly a prism since its faces are not polygons.

rectangle

A 2D shape with four straight sides and four right angles. This means that a square is also a rectangle – a special case in which the four sides are also of equal length – and all other rectangles are **oblongs** (see Fig. 8).

regular polygon

A regular polygon is a closed 2D shape formed of straight lines which has all angles the same size and all sides the same length.

scale factor

When a shape is enlarged or reduced, the scale factor gives the proportional change to the length of each side. For example, a scale factor of 2 means that each side has doubled in length.

side

Used to refer to a boundary or edge of a 2D shape which joins two corners. Initially children may talk about both straight and curved sides; later, 'side' refers only to boundaries of shapes which are straight lines. (See also **edge**.)

solid 3D shape

A three-dimensional (3D) shape, such as a cube, sphere or cone. 'Solid' provides a physical and visual description which helps children distinguish or sort shapes of this type, and hence to develop mental imagery and generalize about them. (See also **flat 2D shape**.)

surface

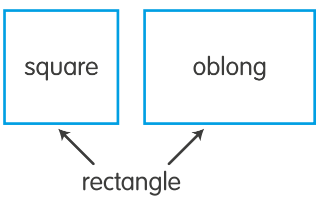
Introduced in Geometry, Measurement and Statistics 1 to refer to the surface of a 3D shape; it may be flat, as in a cube, or curved, as in a sphere. The term 'face', which describes a flat surface only, is introduced in Geometry, Measurement and Statistics 2.

vertex

A point at which the edges of a solid, 3D shape meet. A cube has eight vertices, as shown in Fig. 11.

weight

The heaviness of an object. By weighing an object to find out how heavy it is, we also find out how much 'matter' it contains – that is, what its **mass** is. The standard unit used to measure weight is newtons; the standard units for measuring mass are kilograms and grams.



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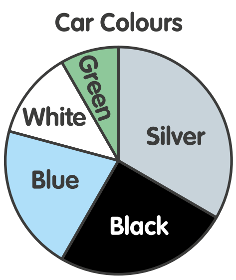

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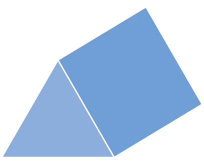

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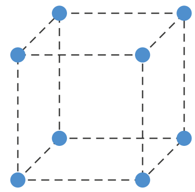

11