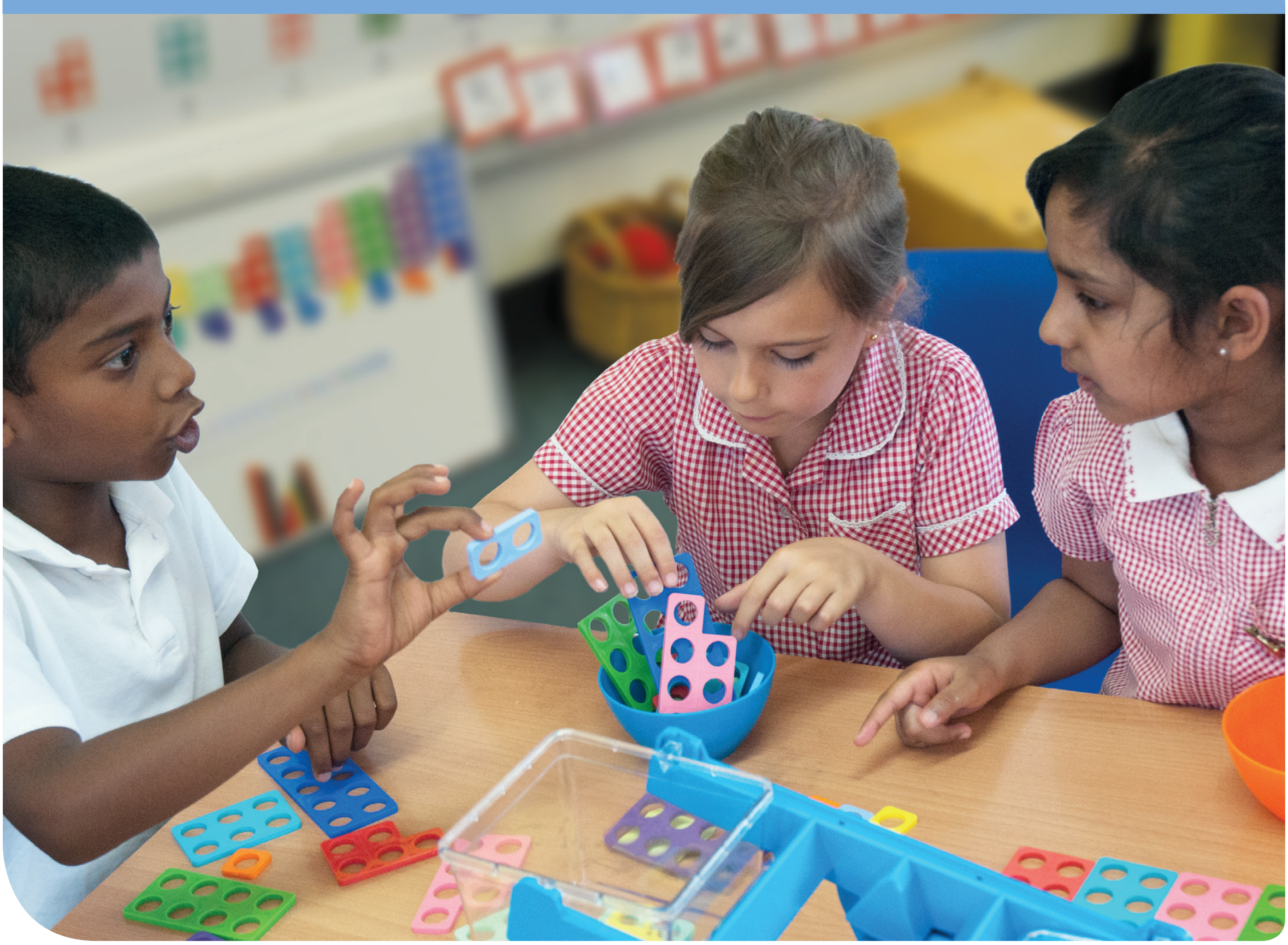
numicon

Number, Pattern and Calculating

Implementation Guide 2



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numicon®

Number, Pattern and Calculating 2 Implementation Guide

Written and developed by
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www.oxfordprimary.co.uk/numicon

About Numicon

Numicon is a distinctive multi-sensory approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships, and generalizing.

Numicon was founded in the daily experience of intelligent children having real difficulty with maths, the frequent underestimation of the complexity of the ideas that young children are asked to face when doing maths and recognition of the importance of maths to them and to society as a whole.

Numicon aims to facilitate children's understanding and enjoyment of maths by using structured imagery that plays to children's strong sense of pattern. This is done through research-based, multi-sensory teaching activities.

Numicon takes into account the complexity of abstract number ideas and seeks to foster the self-belief necessary to achieve in the face of challenge or difficulty.

Through the combination of communicating mathematically (being active, talking and illustrating), exploring relationships and generalizing, children are given the support to structure their experiences: a vital skill for both their mathematical and their overall development.

A multi-sensory approach, particularly one that makes use of structured imagery, provides learners with the opportunity to play to their strengths, thereby releasing their potential to enjoy, understand and achieve in maths. By watching and listening to what children do and say, this enjoyment in achievement is also shared by teachers and parents.

Numicon strives to support teachers' subject knowledge and pedagogy by providing teaching materials, Professional Development and on-going support that will help develop a better understanding of how to encourage all learners in the vital early stages of their own mathematical journey.

Contents

Welcome to Number, Pattern and Calculating 2

Before you start teaching, take some time to familiarize yourself with the Number, Pattern and Calculating 2 starter apparatus pack, the teaching resources and the pupil materials, to help you and the children get the most out of using Numicon.

Use this Implementation Guide:

- to find out more about what Numicon is
- to find out how using Numicon might affect your mathematics teaching
- to learn about the key mathematical ideas children face in the activity groups
- for more detailed information on the theory behind Numicon from Dr Tony Wing.

You will find guidance on how to get the most out of teaching, planning and assessing using Numicon on the Numicon Planning and Assessment Support.

You will find more information, videos, ideas and free resources on the Numicon website: www.oxfordprimary.co.uk/numicon.
Here, you can sign up for our newsletter, which includes suggestions for topical mathematics and updates on Numicon.

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What's in the Numicon starter apparatus pack?

The following list of apparatus supports the teaching of Number, Pattern and Calculating 2. These resources should be used in conjunction with the focus and independent activities described in the activity groups.

Starter apparatus pack contents

- Numicon Shapes – box of 80 (× 2)
- Numicon Coloured Pegs – bag of 80 (× 2)
- Numicon Baseboard (× 6)
- Numicon Feely Bag (× 3)
- Numicon Display Number Line
- Numicon 10s Number Line (× 3)
- Numicon 0–31 Number Line – set of 3 (× 2)
- Numicon 0–100 cm Number Line – set of 3 (× 2)
- Numicon Spinner (× 4)
- Numicon Dice – set of 4
- Numicon 0–100 Numeral Cards (× 2)
- Numicon 1–100 Card Number Track (× 2)
- Numicon Post Box – set of 3
- Number rods – large set
- Numicon Number Rod Trays 1–10 and 20
- Numicon 1–100 cm Number Rod Track (× 3)
- Extra Numicon 10-shapes – bag of 10 (× 2)
- Extra Numicon 1-shapes – bag of 20
- Magnetic strip

Numicon Shapes 1

These offer a tactile and visual illustration of number ideas. The Shapes are also a key feature of the *Numicon Software for the Interactive Whiteboard*, useful for whole-class teaching sessions. However, the Software is not a substitute for children actually handling the Shapes themselves. It is strongly recommended that children are provided with their own individual set of Numicon Shapes 1–10 for use in whole-class sessions.

Numicon Coloured Pegs 2

These red, blue, yellow and green Pegs are useful for making patterns and arrangements as children investigate problems.

Numicon Baseboard

The square Baseboard has 100 raised studs to hold Numicon Shapes and is used in many activities, providing a defined 'field of action'. Uses include: covering the board with Shapes; creating symmetrical designs; matching activities when used with the Numicon Baseboard Overlays; providing a base for building towers with Numicon Shapes and Pegs; making patterns with the Pegs; finding different ways of combining Shapes.

Numicon Feely Bag 3

By feeling for Numicon Shapes in the Feely Bag, children simultaneously visualize the properties of the Shapes, helping them to develop their own mental and tactile imagery of number.

Numicon Display Number Line 4

A central component of number-rich classrooms, the Numicon Display Number Line provides a visual reference for children connecting Numicon Shapes, numerals and number words with the number line.

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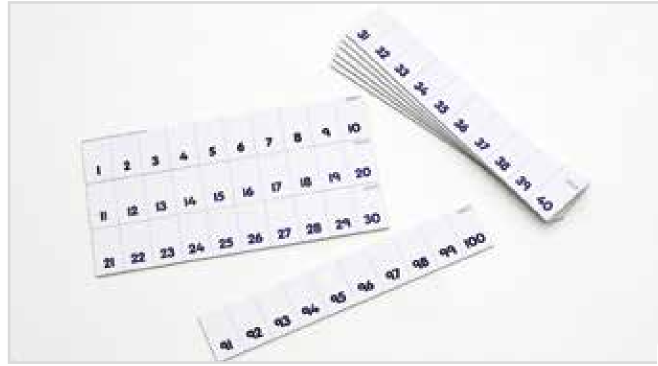
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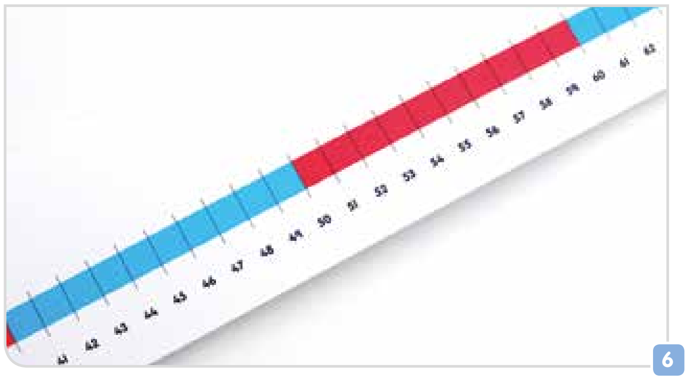
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Numicon 10s Number Line 5

This number line shows Numicon 10-shapes laid horizontally end-to-end and marked with multiples of 10 from 0–100. It helps children to develop a 'feel' for the cardinal value of numbers to 100 and connect this to place value.

Numicon 0–31 Number Line

This number line shows the numerals 0–31, spaced so that children can place a counter on each numeral in independent counting activities, helping them to generalize the idea that the last number in their count tells them 'how many'.

Numicon 0–100 cm Number Line 6

The points on this number line are 1 cm apart and are labelled from 0–100. The number line is divided into decade sections, distinguished alternately in red and blue, to help children find the 10s numbers that are such important signposts when children are looking for other numbers. This resource can also be used with number rods.

Numicon Spinner

The Numicon Spinner can be used in many activities as an alternative to dice. Different overlays (provided as photocopy masters) can be placed on the spinner to generate a variety of instructions for children to follow, including: numerals, Numicon Shape patterns, coin values and symbols of arithmetic notation. The spinner also features on the *Numicon Software for the Interactive Whiteboard*.

Numicon Dice

A set of four 22 mm dice, featuring Numicon Shape patterns alongside the numerals: two 0–5 Dice, one 5–10 Dice and one +/– Dice. These are useful for many of the independent practice activities as an alternative to using spinners.

Numicon 0–100 Numeral Cards

This pack of 0–100 numeral cards can be used in several activities for generating numbers which children then work with. It is also used in some of the whole-class and independent practice activities and games.

Numicon 1–100 Card Number Track 7

This number track is divided into ten strips, numbered 1–10, 11–20, 21–30 and so on. The sections can be arranged horizontally end-to-end as a number track, or as an array similar to a 100-square.

Numicon Post Box

A card post box through which Numicon Shapes and cards, bearing a calculation, can be posted. This resource helps to make practice activities engaging and playful.

Number rods 8

A box of number rods contains multiple sets of ten coloured rods, 1 cm square in cross section. The shortest is 1 cm in length, the longest 10 cm. These offer another structured illustration for number and are used alongside Numicon Shapes in many of the activities. Being centimetre-scaled, they can also be placed along the Numicon 0–100 cm Number Line.


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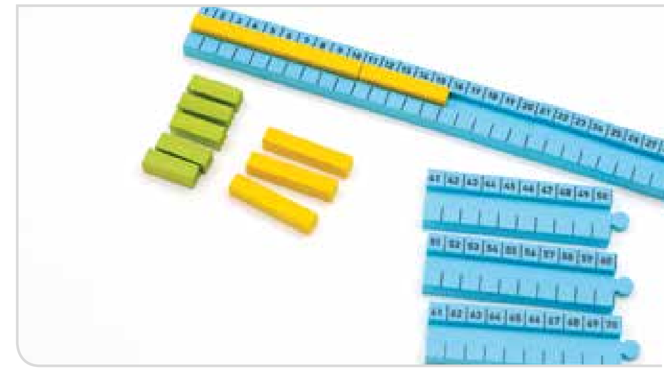
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Numicon Number Rod Trays 1–10 and 20 9

This set comprises a Number Rod Tray for each number up to 10, plus one for 20. They are useful for building up patterns and for number fact work with the number rods.

Numicon 1–100 cm Number Rod Track 10

Use this for teaching about place value, partitioning, multiplying and dividing. The decade sections click together into a metre-long track. Designed to take number rods, it can be separated easily into sections and arranged as an array.

Magnetic strip

This self-adhesive magnetic strip can be cut into pieces and stuck onto the back of Numicon Shapes or number rods so that they can be used on a magnetic whiteboard.

Available separately

Numicon Software for the Interactive Whiteboard 11

This rich interactive tool is designed for use with the whole class to introduce key mathematical ideas. It includes: number lines featuring Numicon Shapes, the Numicon Pan Balance, objects for counting, coins, Numicon Spinners and much more.

Individual sets of Numicon Shapes 1–10

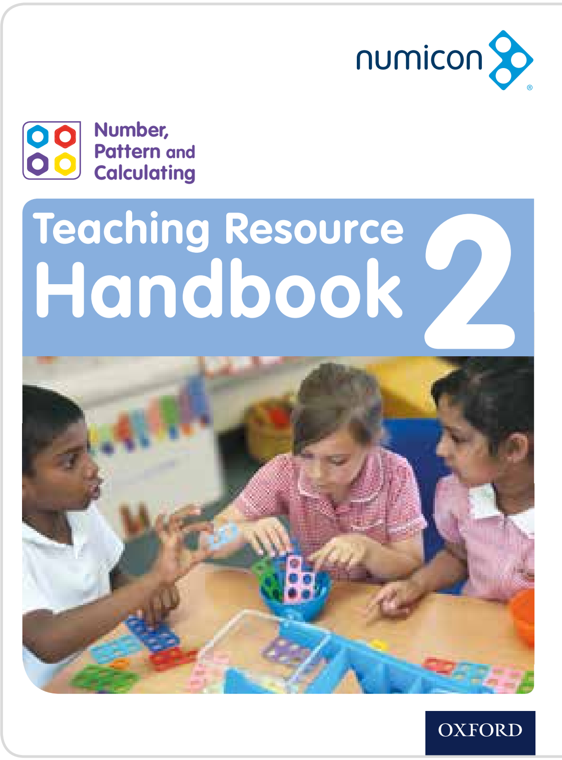
These are designed for multi-sensory whole-class lessons, where each child has their own set of Shapes and is encouraged to engage with them. They are especially useful when used in conjunction with the *Numicon Software for the Interactive Whiteboard* to help teachers assess children's individual responses from the Shapes children hold up.

Numicon Pan Balance 12

Using Numicon Shapes or number rods in this adjustable Pan Balance enables children to see equivalent combinations, helping them to understand that the '=' symbol means 'is of equal value', thus avoiding the misunderstanding that it is an instruction to do something. Children can easily see which Shapes are in the transparent pans. A virtual balance is also featured on the *Numicon Software for the Interactive Whiteboard.*

Other equipment

Some activities use apparatus found in most classrooms, e.g. **sorting equipment, base-ten apparatus** and **interlocking cubes**. Opportunities to use these are highlighted in the '**Have ready**' sections of each focus and independent practice activity.



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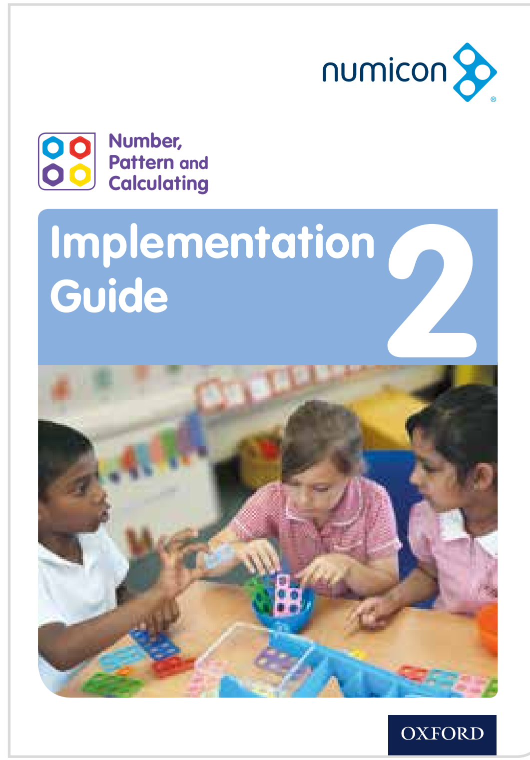

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What's in the Numicon teaching resources?

Number, Pattern and Calculating 2 Teaching Resource Handbook 13

This contains 30 activity groups clearly set out and supported by illustrations. Each activity group begins with the educational context, learning opportunities, assessment opportunities and important mathematical vocabulary that support mathematical communicating in the activity group. To support teachers' assessing of children, there are notes on what to 'look and listen for' as children work on the activities, as well as suggestions for whole-class and independent practice. Photocopy masters supporting the 30 activity groups are included at the back of the Teaching Resource Handbook.

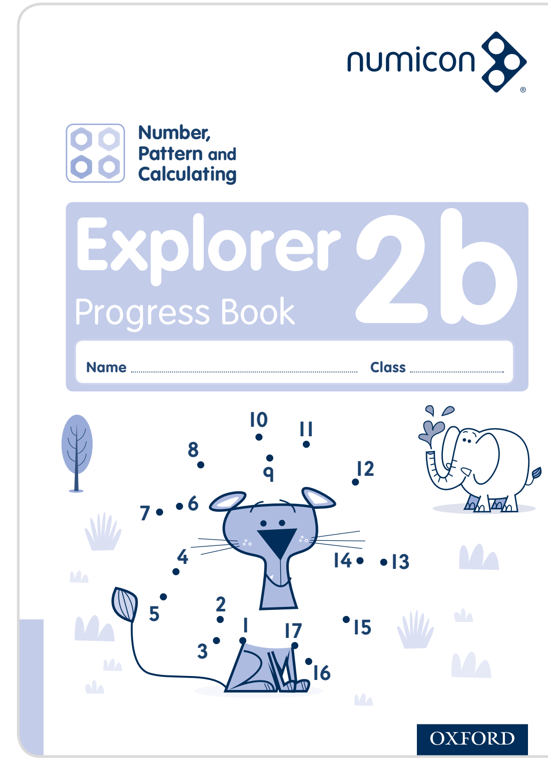
Support for planning and assessment is included at the front of the handbook. There you will find:

- information on how to use the Numicon teaching materials and the physical resources
- long- and medium-term planning charts that show the recommended progression through the 30 activity groups
- milestones to help assess how children are progressing in their learning
- an overview of the activity groups.

Number, Pattern and Calculating 2 Implementation Guide 14

This introduces what Numicon is and how Numicon helps children meet the demands of learning mathematics.
It also includes some practical advice about what to do when preparing to teach with Numicon and answers some key questions about how to use Numicon in practice.
The 'Key mathematical ideas' section provides useful explanations about the important mathematical ideas children will meet in the 30 activity groups of the Teaching Resource Handbook and how these ideas develop when teaching with Numicon. There is also a further chapter with more background detail on the research that inspired Numicon and the rationale behind the pedagogy.

The different sections of the Implementation Guide can be accessed as and when necessary to best help you with your teaching.



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Number, Pattern and Calculating 2 Explorer Progress Books 15

The Explorer Progress Books offer children the chance to try out the mathematics they have been learning in each activity group. In children's responses, teachers will be able to assess what kind of progress individual children are making with the central ideas involved in each activity group.

It should be stressed that the challenges in the Explorer Progress Books are not tests. There are no pass/fail criteria; the challenges are simply designed to reveal how well children can use the mathematics they have been learning in a new situation.

Being able to use mathematics in an unfamiliar situation is a significant indicator of children's understanding. Many of the tasks from the Explorer Progress Books set mathematics in a new or different context and, where possible, the challenges are set in an open way, inviting children to show how they can reason with the ideas involved rather than testing whether they have learned a routine solution to routine tasks.

As with children's classroom activity work, a range of apparatus should be freely available to children as they reason with the ideas in their Explorer Progress Books.

In addition, there is also scope for self-assessment in each Explorer Progress Book in the form of a Learning Log, which can be used flexibly throughout a term, or to summarize the learning of a block of work.

Each Explorer Progress Book broadly covers a 10–12 week period, so three are provided to cover the academic year.

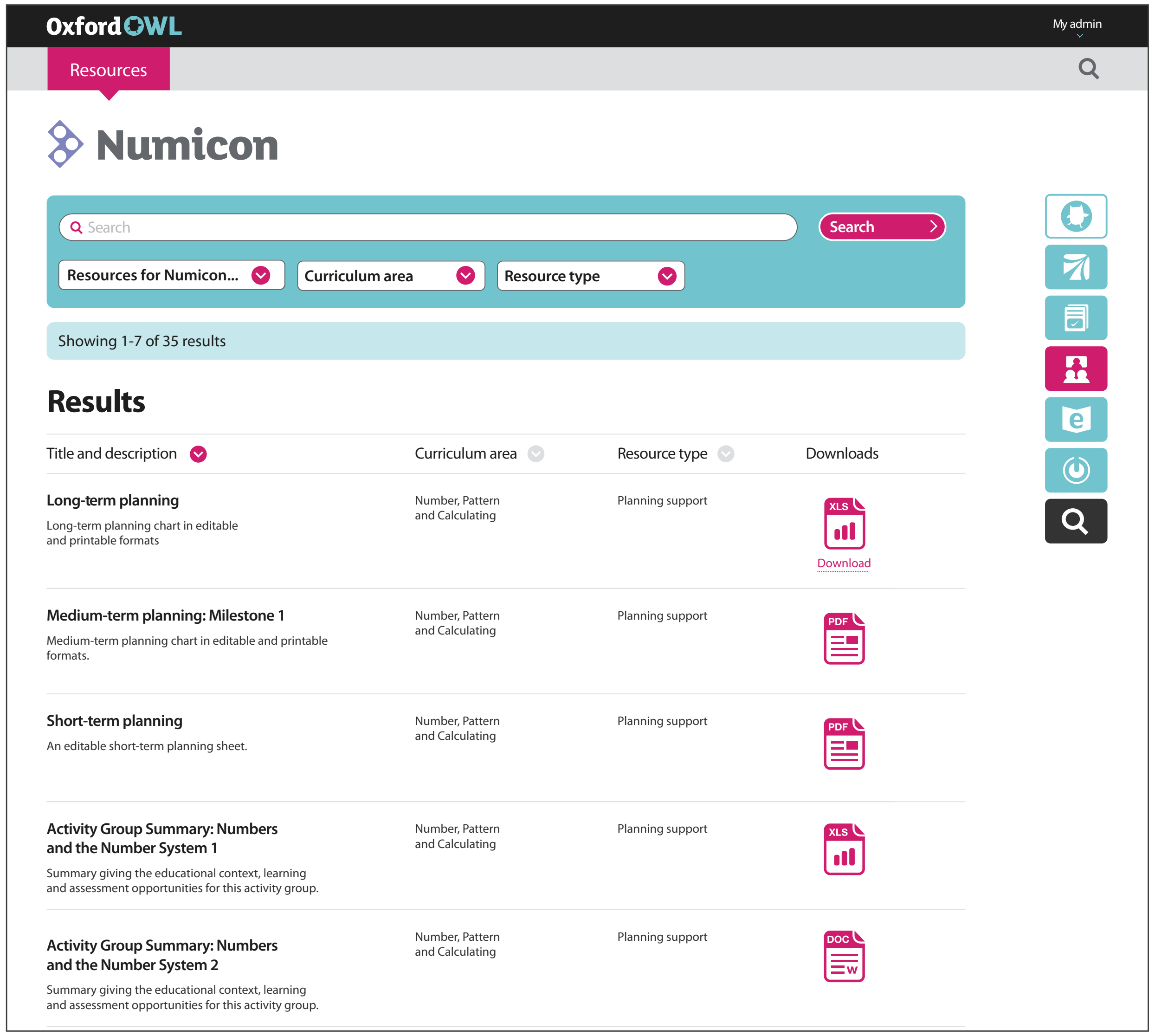
Number, Pattern and Calculating 2 Explore More Copymasters 16

The Explore More Copymasters offer children the chance to practise and discuss mathematics at home with a parent or carer.

An activity has been included for each activity group so that children have ongoing opportunities to practise their mathematics learning outside of the classroom.

Each activity is supported by information for the parent or carer on the mathematics that has been learned in class beforehand, and the learning point of the activity itself. Guidance on how to complete the activity is included, as well as suggestions for how to make the activity more challenging or how to develop the activity further in a real-life situation.

The Explore More Copymasters can be given to an adult or child as a photocopied resource.



Numicon Planning and Assessment Support

The Numicon Planning and Assessment Support is designed to be used flexibly within schools' own planning formats.

Within the support, you will find a range of resources including short videos introducing Number, Pattern and Calculating 2 and offering advice that will help you get started teaching with Numicon.

There is also an editable summary of each activity group, including the title and number of the group, the educational context, the learning opportunities, the mathematical words and terms to be used with children as they work on the activities, and the assessment opportunities. There are suggestions for how to use these activity group summaries in your planning, as well as editable short-, medium- and long-term planning frames.

Assessment grids that support monitoring of children's work on the Explorer Progress Books and editable versions of the milestones for assessing chidren's progress are also available.

Charts that map Numicon to the English, Welsh, Scottish and Northern Irish curricula have been included in these resources, as have charts showing the progression of the Numicon teaching programme across Number, Pattern and Calculating and Geometry, Measurement and Statistics.

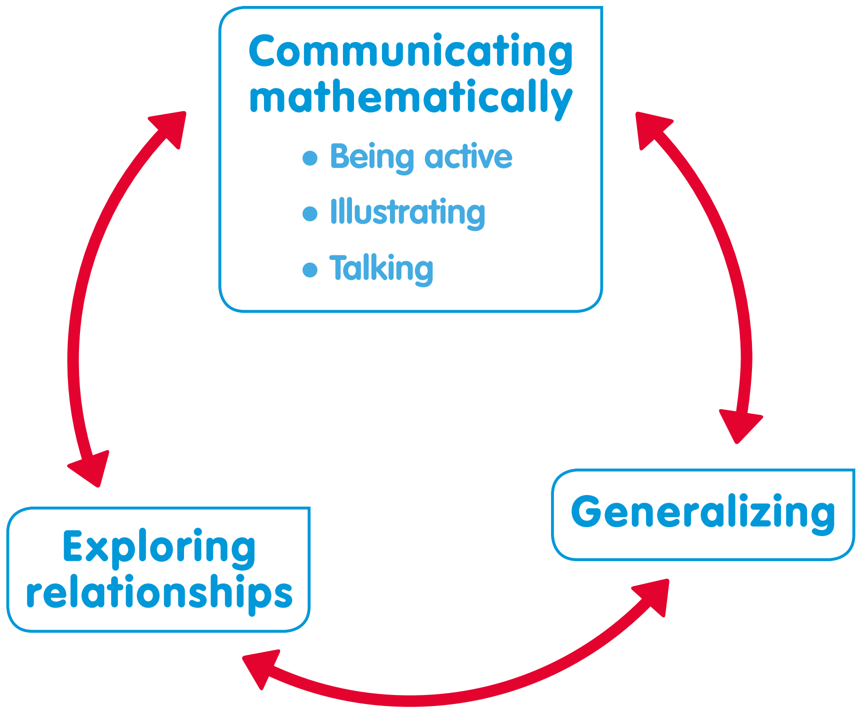
Printable versions of all the photocopy masters and resources for creating mathematics displays in your classroom are also provided.

What is Numicon?

In order to illustrate how using Numicon in your teaching can help children learn mathematics, this section looks at:

- what Numicon is
- what doing mathematics demands of children
- how using Numicon helps children meet these unique demands.

If you would like further information on the theory behind Numicon from Dr Tony Wing, please turn to page 60.



What is Numicon?

Numicon is a distinctive approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships, and generalizing.

Communicating mathematically

Doing mathematics involves communicating and thinking mathematically – and these are two sides of the same coin. We think in the same ways that we communicate, and communicate in the same ways that we think. As children learn to communicate mathematically, they learn to think mathematically. This involves them in the following:

Being active: Teaching and learning with Numicon requires children to be active. This does not only mean being physically active (e.g. fitting physical objects together, finding a number on a number line, drawing a shape), but is a requirement reflecting the understanding that mathematics itself is activity: mathematics is something children are learning to *do*.

What this means in practice is that it is always the children themselves who are to do the mathematics. Telling children (or showing, or explaining) 'what to do' can encourage children to be passive. Numicon asks that the children do the mathematics (i.e. both the activity and the thinking). In other words, children actively learn to do mathematics for themselves.

Illustrating: Doing mathematics (i.e. thinking and communicating mathematically) necessarily involves illustrating, because mathematics is about studying relationships between objects, actions and measures, and it is impossible to explore such relationships without some kind of spatial imagery being involved. Even when thinking about just two numbers, one of them 'comes before' the other one on a number line or is said to be 'bigger' or 'higher'; these relations all involve spatial imagery.

Numicon explicitly involves illustrating in every activity. This is how relationships in a wide variety of contexts are both explored and communicated.

Talking: Since doing mathematics involves communicating mathematically (both with others and with ourselves), doing mathematics also involves talking. Talking is an essential aspect of all Numicon activity.

Throughout Numicon, talking means dialogue in which points of view are exchanged between teachers and children, and between children and children. All partners in dialogue will be actively involved, not just passively hearing, or waiting to speak. Such exchanges encourage reflective thinking as children learn to discuss different points of view, both with others and with themselves.

Exploring relationships (in a variety of contexts)

Doing mathematics involves **exploring relationships** (i.e. the structure) in any situation in order to develop some kind of control in that area. Relationships can be between amounts, between positions, between shapes, between things that are varying, between things that are constant, or between any combination of all or any of these things. Fully understanding the relationships in a situation makes it predictable, thus making it possible to find out and/or manage what is needed. The mathematical reasoning children are asked to do is their expression of relationships they perceive.

Numicon ensures that children explore the relationships within a wide range of contexts so that they learn not only *how* to do mathematics, but *when* the mathematics they are learning is useful.

Generalizing

In doing mathematics, exploring relationships and looking for patterns in various situations lead to **generalizing**. It is generalizing that allows us to make whole ranges of new situations predictable.

Numbers are generalizations that we all use to make predictions when calculating. For example, the '6', '2' and '8' in the number sentence '6 + 2 = 8' are generalizations; 6 of anything and 2 of anything will together always make 8 things, whatever they are.

'The angles of a triangle add up to 180°' is a generalization that is often used when doing geometry; 'the area of a circle is πr^2' is another that is used when measuring.

In each of these cases, noticing patterns in relationships allows us to generalize about an infinite number of other, similar situations.

It is because we make and use generalizations continually as we do mathematics that mathematical thinking and communicating will appear to have an abstract character for children if they are not involved in doing the generalizing for themselves.

Communicating mathematically, **exploring relationships** and **generalizing** all come together when *doing* mathematics.

What doing mathematics demands of children

In learning to do mathematics at the age they are asked to, children face a unique challenge in their school curriculum: thinking and communicating about abstract objects.

Pure numbers, such as 6 and 254, are abstract objects and in no other subject are children asked to work with and reason about such things so early. It is not surprising that children can hesitate, be puzzled, get stuck, or take time to work things out for themselves.

Most mathematicians will say that doing mathematics is about looking for patterns in situations. It is, but there is more to it than that. When you find a pattern, it means you have noticed something regular, something that always seems to be happening, and this means that any pattern you see is a generalization of yours. Mathematics deals almost entirely in generalizations.

Which is what numbers are: as generalizations, they are abstract objects. Very early on in mathematics, young children are asked to do things with lots of abstract objects, like those we call '3', and 'ten'. Not '3 pens', or 'ten sweets', or '3 friends'. Just '3', or, even more curiously, the two-digit '10' by itself. Notice how quickly children are asked to add and subtract these abstract objects to and from each other. There are high expectations of every child from the very beginning in mathematics.

The central problem: communicating with, and about, abstract objects

How is it possible to communicate about abstract objects? And, since thinking is communicating with ourselves, how is it possible to *think* about abstract objects?

Notice that 'abstract' does not mean 'imaginary'. We can easily picture imaginary things, such as unicorns and beings from outer space, but abstract things are different. Abstract things include qualities like 'truth', 'intelligence', 'common sense', as well as generalizations such as '6 of anything'. The problem is, as soon as you try to picture '6 of anything', you find you are imagining '6 of something'.

The answer, as Jerome Bruner observed, is that we think about abstract objects with symbols; in the case of numbers, we do it with numerals.

The important thing about symbols is that they do not attempt to show literally what is being talked about; they are simply arbitrary, conventional marks on a page (or spoken words). When what we are communicating about with these symbols is abstract, that absence of a picture is inevitable; how could we possibly picture something that is abstract?

It is easy to picture '3 pens' or 'ten friends', but what might the abstract '3' look like? Or, how about the curious two-digit '10'? Since numerals do not look like the abstract things they 'stand for', how are children to learn to interpret such arbitrary symbols in their thinking and communicating? Doing mathematics – thinking and communicating about abstract things with symbols – is certainly not easy for young children.

How does Numicon help?

Essentially, Numicon does two things. Firstly, Numicon acknowledges that in order to understand what numbers are, children have to generalize. Secondly, Numicon follows Bruner's advice in using children's actions and imagery to prepare for their use of mathematical symbols in their thinking and communicating. In practice, these two things are done at the same time.

In Bruner's terms, *enactive* and *iconic* representations (action and imagery) are used to inform children's interpretation of the *symbolic* representation (e.g. numerals) that is necessary for communicating their pattern spotting (generalizing). To aid children's necessary generalizing, strong focus is placed upon the use of structured materials.

Generalizing and reasoning – an early years example of teaching with Numicon

Initially, in common with most teaching, Numicon involves a wide variety of everyday objects (such as beads, cubes, pegs and counters, see Fig 1) in order to help children develop their counting, before then introducing the challenges of calculating.

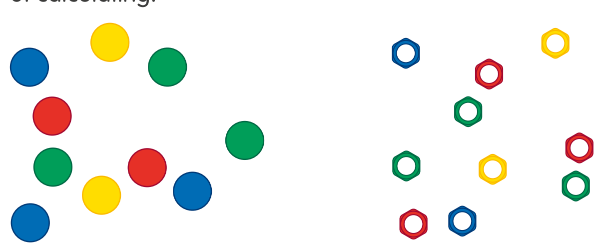
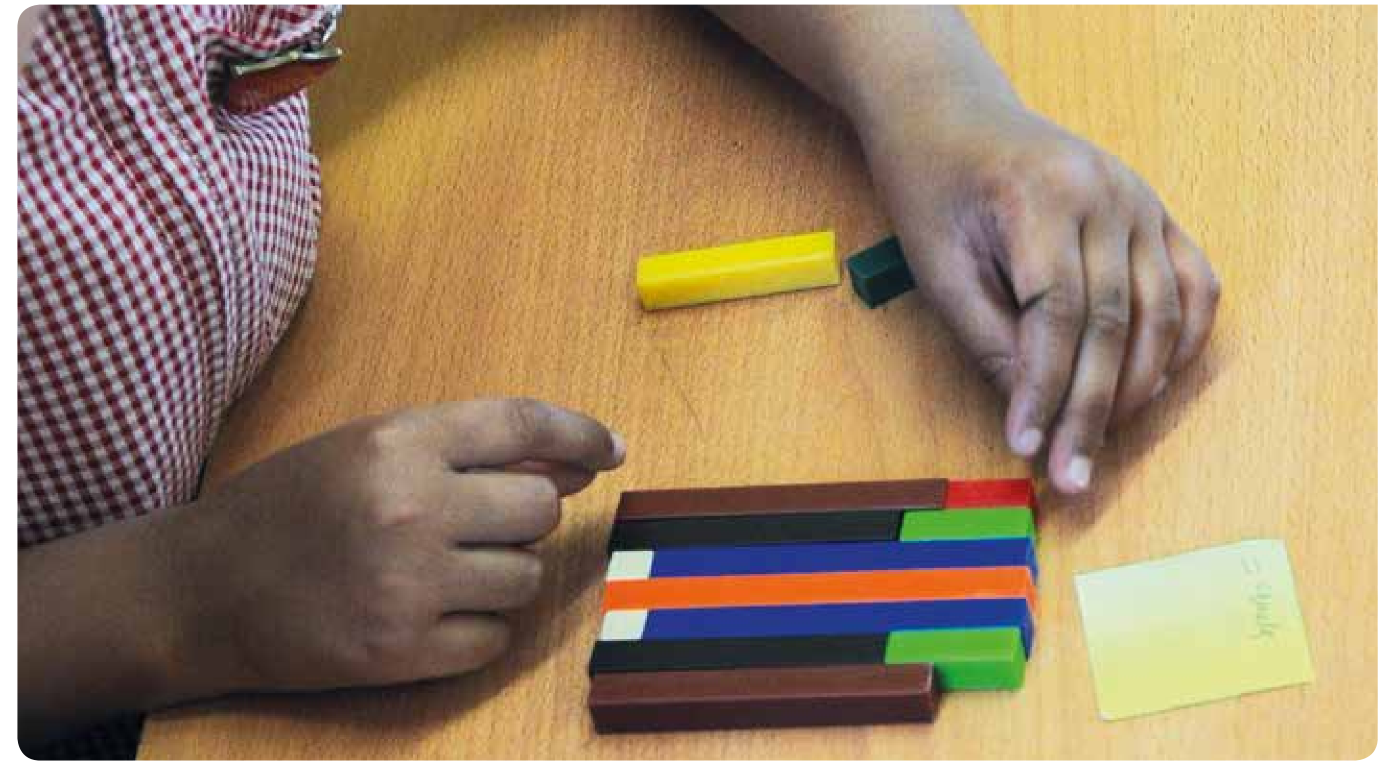


Fig 1



Importantly, Numicon also introduces sets of structured materials in which individual pieces have *regular* physical relationships with each other, for example, Numicon Shapes and number rods, see Fig 2. Children explore the physical relationships within these structured materials by, for example, ordering pieces, comparing them, combining them physically to make others.

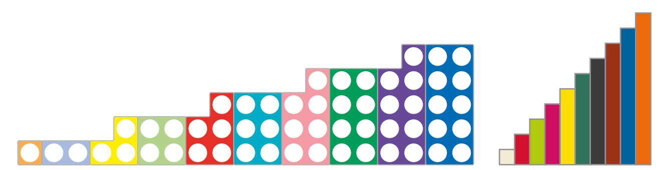


Fig 2

Thus, as children work with loose collections (beads, cubes) and with structured materials (Numicon Shapes and number rods), they are able to combine being active with physical objects and images as illustrations in their talking and thinking about numbers of things.

Numerals are introduced in association with loose collections of objects and with structured materials as children work; thus, symbolic representation appears alongside children's ongoing action and imagery with physical objects.

Number lines (e.g. Fig 3) are introduced to further illustrate the ordering that is evident within the structured materials, and to reinforce the associated ordering of numerals.

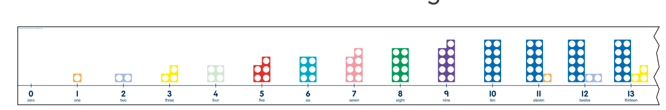


Fig 3

Importantly, loose collections of objects are arranged into the regular patterns of the Numicon Shapes; individual number rods are found to be equivalent in length to different multiples of the smallest 'unit' rod (e.g. Fig 4).

Fig 4

Through doing these activities, children learn that *any* collection of loose objects can be arranged into Numicon Shape patterns that can then be 'read'. Children learn to 'see' how many objects are in a collection without counting; numbers of things begin to 'take shape' visually, in organized ways.

It is now possible to prepare children for further generalizing about numbers through exploring relationships between numbers of things. Children's mathematical thinking and communicating continue to develop through their being active with the objects they are given, and in illustrating their talking about what they see and do.

Children generalize that any collection of loose objects can be arranged into Numicon Shape patterns. They also generalize that any number of 'unit' number rods may be exchanged for (will be 'as long as') one or more of the larger number rods. Thus *any* number of loose objects can be converted into (is equivalent to) one or more Numicon Shapes or number rods.

In realizing that any collection of loose objects can be arranged into Numicon Shape patterns, and that any number of unit

cubes is equivalent to one or more number rods, children become able to 'see the general' in any particular illustration; Numicon Shapes and number rods can be used to illustrate, in an organized way, any numbers of any kinds of things.

Numicon Shapes and number rods themselves may now be used to explore and to communicate about number relationships in general. They have become communication mediators in discussions about numbers and their relationships.

As a particular example, the Numicon Shape that has three holes fits together physically with the Shape that has five holes. The result 'forms the same shape as' (is equivalent to) the Shape that has eight holes, see Fig 5.

Similarly, the number rod worth three units, combined end-to-end with the rod worth five units, are together as long as the rod worth eight units, see Fig 6.

When laid end-to-end along a number line or number track, the '3 rod' and the '5 rod' together reach the position marked '8' on the line.

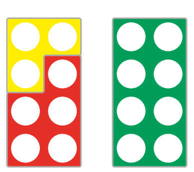


Fig 5

Fig 6

From these actions, and with these illustrations, children are able to generalize that: three *anythings* and five *anythings* together will *always* make eight things.

This generalization can be expressed using the numeric and verbal symbols: 3 and 5 together are equivalent to 8.

Later on, using further actions and further illustrations, children become able to interpret and use the further symbols '+' and '=' to express their generalization thus:

3 + 5 = 8

Importantly, at this stage children will have begun to use number words (one, two, three) as *nouns* instead of as adjectives (two sweets, three pencils) in their talking.

With their use of number words as nouns, numbers as *abstract objects* have now appeared in children's mathematical thinking and communicating, referred to with *symbols*.

Such generalizing and use of symbols can now be exploited further. If 'three of *anything*' and 'five of *anything*' together always make 'eight *things*', then:

3 tens + 5 tens	=	8 tens
3 hundreds + 5 hundreds	=	8 hundreds
3 millions + 5 millions	=	8 millions

or

30 + 50	=	80
300 + 500	=	800
3,000,000 + 5,000,000	=	8,000,000

Such is the power of generalizations, and of the symbolic notation that children can by this stage use to think and communicate mathematically about them.

Progressing from such early beginnings

The foregoing example also illustrates how Numicon supports the teaching of children's subsequent mathematics.

In a wide variety of contexts, children are offered opportunities to be active and to illustrate their talking about any (and all) relationships they are exploring between both physical and abstract objects. They do this in order that they may generalize and, as a consequence, think and communicate about their generalizations with the conventional symbols of mathematics: in other words, that they may *do* mathematics.

Of course, there is much more beyond this first example. Children have many other relationships to explore, further generalizing to do, and thus many more conventional symbols to learn to interpret and use. However, wherever it is used, the approach is essentially the same. *Enactive* and *iconic* representation are used to inform children's interpretation of the *symbolic* representation that is necessary for thinking and communicating about their constantly developing pattern spotting (generalizing).

The going gets harder as children progress in their number work: place value, ratios, fractions, decimals and negative numbers will all challenge children's capacity to generalize onwards from these early beginnings. However, such advanced work is equally possible with the same approach of actions and illustrations building towards generalizing, and a consequent use of symbols to think and communicate about the generalized, abstract mathematical objects created. This is how the symbolism of mathematics becomes meaningful.

In their work on geometry, the generalizations that children reach in their mathematical thinking and communicating happen more gradually as they progress towards being able to reason about invented abstract objects such as 'any triangle' and (later) 'any polygon'.

It is impossible to draw the abstract object 'any triangle' in the same way that it is impossible to imagine 'six of anything'. As soon as you draw a triangle, however you have chosen to draw it, you have drawn a particular one; it does not matter whether you draw one that is right-angled, isosceles, equilateral, or scalene, what you have drawn is not a general triangle – it is a particular one, e.g. Fig 7.

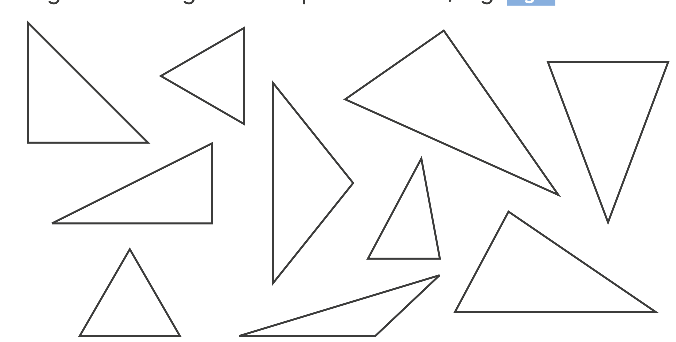


Fig 7

However, as with generalizing about numbers, in doing geometry, much exploring of relationships with action and imagery (enactive and iconic representation) prepares children for reasoning meaningfully about 'any triangle' with symbolic representation (words and symbols).

In this example, as children physically construct and transform many varieties of triangles with dynamic materials, what they see before their eyes is how lengths of sides may vary endlessly as angles change, how angles may vary endlessly as lengths change and yet, despite all the variety they see, the shapes they are creating all turn out to be triangles.

Dynamic mental imagery deriving from these physical experiences then allows children to speak and to think of 'any triangle' as they imagine an infinitely flexible closed 2D shape with three straight sides.

By stressing the straightness of the sides, that there are always exactly three sides and that the sides fit together to 'close' the shapes they make, and by ignoring the constantly changing lengths of sides and sizes of angles, children become able to imagine 'any triangle' and to reason about it with words and symbols.

Then, having generalized to a stage when their mental imagery allows them to imagine and speak meaningfully about 'any triangle', children are in a position to further generalize that, for example, 'the angles of any triangle add up to 180°' through further action, imagery and reasoning with symbols.

In the same way that generalizing about numbers is approached, exploring relationships with action and imagery in geometry prepares children for reasoning meaningfully about abstract mathematical objects with symbols.

We reach all the generalizations of mathematics, and the symbols we use to express them, through being active and illustrating our talking about the relationships we are exploring.

Doing mathematics in the world – solving problems

Of course, *doing* mathematics in the everyday world is not simply about making generalizations and using symbols. Crucially, it also involves making *use* of generalizations to solve problems in particular situations.

For example, the generalization '4 x 25 = 100' allows us to predict that the perimeter of a square of side 25 cm will be 100 cm, that the area of a field measuring 4 m by 25 m will be 100 m^2, and that if you save £25 a week for four weeks you will have £100. It can also be very useful to help calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

Children need to be able to readily connect the generalizing that they do with mathematical symbols with everyday world situations in which those generalizations are useful.

In their more general problem solving, children learning, for example, *how* to divide one number by another, need to learn *when* that operation is useful.

The Numicon teaching materials organize activities into groups based around mathematical themes. For example, in the *Number, Pattern and Calculating 2 Teaching Resource Handbook*, 'Introducing fractions as numbers' (Numbers and the Number System 6) and 'Partitioning into tens and ones to answer adding and subtracting problems' (Calculating 6) each activity group is introduced with, or involves a context in which that mathematics is useful.

The activities in 'Introducing fractions as numbers' begin by asking children to think of all the situations in which they use the term 'halfway' and use these contexts to develop discussion of positions 'between' two whole numbers along a number line. The activities from 'Partitioning into tens and ones to answer adding and subtracting problems' begin with a context in which parties of schoolchildren have to fit themselves into 10-seater carriages during a theme park trip. In both activity groups, it is clear to children that the situations are familiar and relevant to their experiences.

In the *Geometry, Measurement and Statistics 2 Teaching Resource Handbook,* a group of activities on symmetry encourage children to discuss reflective symmetry in living creatures, buildings, leaves and mirrors.

In these ways, children's mathematics is introduced, wherever possible, within contexts in which that mathematics is useful. Such contexts help children to 'see the point' of the mathematics they are learning, and prevent mathematics becoming a series of answers to problems they have never had.

Flexibility, fluency, and persistence
Being able to do mathematics effectively also includes being able to remember basic things such as those generalizations known as 'times tables' and number 'facts'. There are many practice activities built into the Numicon written materials that encourage children to develop a regular familiarity with basic facts.

Even more important to effective functioning is flexibility in mathematical thinking and communicating. Three kinds of flexibility are especially useful.

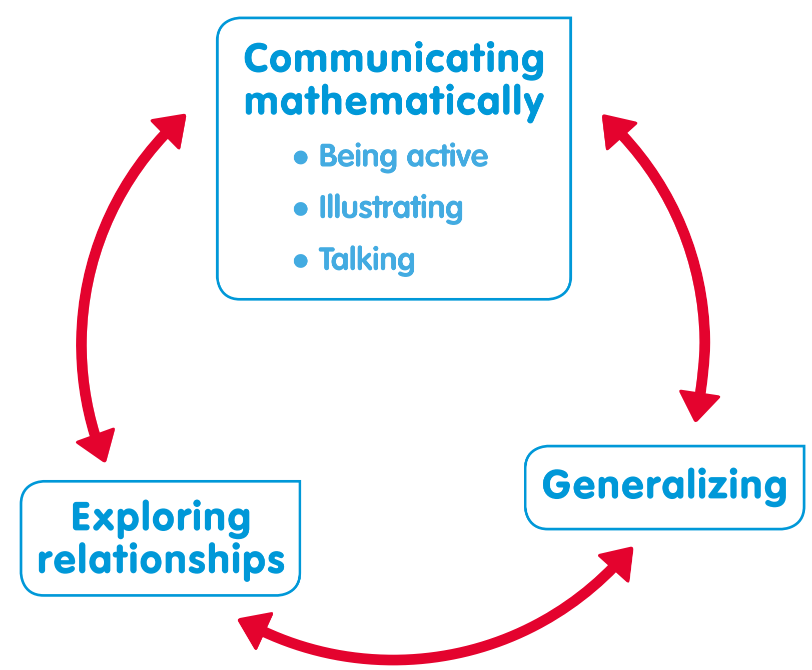
Through being active, children are able to 'invert' their actions; put more simply, this is about 'doing and undoing'. As children have found 'how many' objects there are in a collection by physically grouping them into tens (and hundreds), they are later readily able to 'partition' numbers ('undo' their groupings) when calculating with symbols. As children can combine and separate Numicon Shapes and number rods physically ('do' and 'undo' their actions), they are able to connect adding and subtracting as inverse operations and to check a subtracting calculation by adding.

The introducing and encouraging of a variety of ways of calculating means that children are able to choose methods of calculating that suit the particular numbers involved, rather than adopting a standard method for any calculation. Who would want to subtract 1998 from 4673 using a column method, just because the numbers are big?

As Numicon approaches children's use of mathematical symbols with prior actions and imagery, if children become 'stuck' or hesitant while working with symbols, it is always possible for them to recall and return to the supporting activity and illustration from which their original generalizing arose. For example, if they are working on prime numbers, using symbols, they can immediately return to actively illustrating how factors 'go into' their various multiples physically with Numicon Shapes and with number rods. They can follow a similar pattern with any other kind of number relationship. This flexibility is one of movement backwards and forwards between Bruner's *enactive*, *iconic* and *symbolic* forms of representation in their thinking and communicating.

Finally, because Numicon continuously emphasizes communicating mathematically, it is possible to stress to children that all is not lost when they feel 'stuck'. The thing to do, always, when they 'don't know what to do' in a situation is to communicate.

Being active, illustrating and talking about the relationships we are exploring *is* doing mathematics. Persistence – an invaluable quality when doing mathematics – comes from continuing to communicate, with yourself and/or with others, whenever (for the moment) you cannot see where you are going.



Preparing to teach with Numicon

This section is designed to support you with practical suggestions in response to questions about how to get started with Numicon in your daily mathematics teaching. It also contains useful suggestions on how to plan using the long- and medium-term planning charts as well as information on how to assess children's progress using the Numicon materials.

When getting started with Numicon, as well as reading this section, it can be helpful to refer to the suggested teaching progression in the programme of Numicon activities. This, along with the long- and medium- term planning charts, can be found in the *Number, Pattern and Calculating 2 Teaching Resource Handbook.*

In this section you will find overviews of:

How might using Numicon affect my mathematics teaching?

There will be a constant emphasis on encouraging children's mathematical communicating, especially when the going is tough. This is because the mathematical communicating *is* the 'doing' of maths. The increased focus on mathematical communicating will make it easier to judge whether children are facing a suitable level of challenge through watching, questioning and listening to them as they work through activities. Numicon aims to encourage children to relish challenge, have a sense of achievement when a challenge is overcome and be excited about and ready for the next one.

As the focus on mathematical communicating grows, you may become more aware of the words and terms you use in your teaching. It is important to use words and terms consistently. Try to encourage other adults in your class – and throughout the school – to illustrate the consistent use of mathematical language.

Consider the imagery being used in the classroom. Listen for children using these same words and imagery to explain their ideas, although to start with they may use them only hesitantly. As communicating becomes established as part of the culture of the classroom, children will increasingly join in conversations with you and their classmates and maths lessons will develop into dialogue with ready use of structured imagery to illustrate ideas.

A sense of shared endeavour will develop in the class as children persist through difficulty to solve problems. So, when children know they are 'stuck', they will know that the thing to do is to talk about it, to try to explain what the difficulty seems to be and to use illustrations and actions to express the problems. Careful questioning can encourage children to really think about 'difficulty' and to naturally come to solutions.

How can I encourage communicating?

Children respond to the examples around them. As such, the ways in which you communicate mathematically provide a model for children's communicating. Engaging in dialogue with children, actively listening to what they are saying and responding sensitively with thoughtful questions, will encourage children to listen to one another and respect each other's ideas.

Make sure that structured apparatus (e.g. Numicon Shapes, number rods, base-ten apparatus) and other communication mediators (e.g. counters, Unifix cubes) are freely accessible. Observe how children use these, listen to what they are saying, watch what they are doing and respond with questions and qualified praise when you notice active listening and thoughtful questioning. What children do and what they say will help you to understand what they are thinking.

The ways in which children are grouped, or paired, for working together has an impact on their communicating, so this needs careful consideration.

Using daily routines to encourage communicating

Introducing a 'morning maths meeting' has proven to be very successful in encouraging children's mathematical communicating. The 'morning maths meetings' are about 15 minutes long and are usually held at the start of each day.

These meetings are oral and practical and include a small selection of key routines every day in which children practise rapid recall and gain fluency with the ideas and number facts that are the focus for that term. These will include: counting practice (e.g. counting in steps of 2, 3, 5 and 10, counting on or back from a number within the broad counting range for the class, counting along a fraction number line); rehearsing number facts (e.g. adding and subtracting facts to 10, adding and subtracting multiples of 10, practising multiplying facts for the 2, 3, 5 and 10 times tables); discussing observations about a mathematically-rich object that you or a child has brought in (e.g. a 'sale' price tag, a book of stamps or an interesting-shaped container); solving a mathematical problem that has come up in the class, in the school, in the news, at home, or in a story.

You can refer to some of the whole-class practice activities from the activity groups in the Teaching Resource Handbook for ideas for 'morning maths meetings'. You could also use one of the focus activities from the activity groups for problem solving. You might also want to keep track of the number of school days in the school year and hold a celebration on the 100th day.

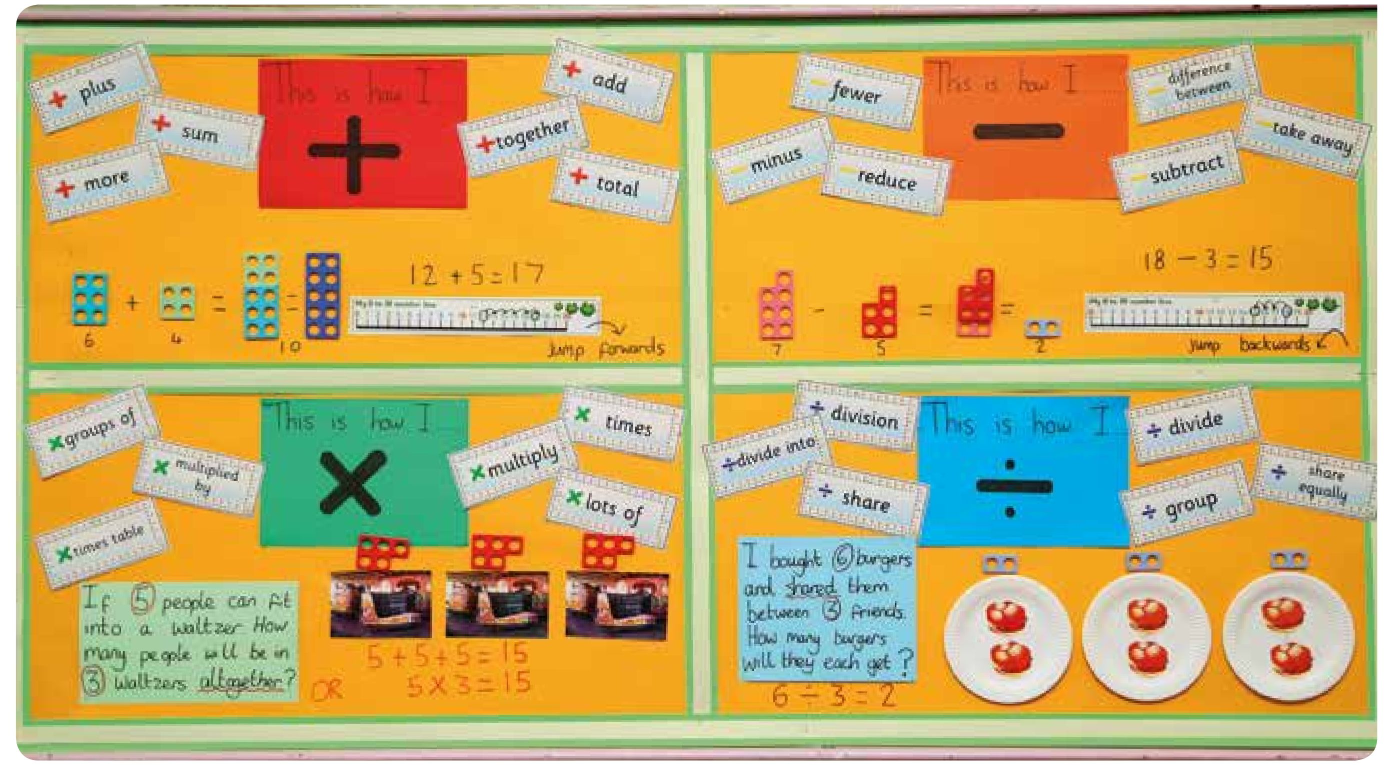
Children's mathematical thinking and reasoning can be encouraged at different times during the day. For instance, when children need to stand in line before going out to play or going to assembly, provide them with statements that will require them to think carefully. You could say: 'If your birthday is in the first month of the year, please join the line'. Vary with other suitable statements that will test children's reasoning, e.g. 'Join the line if your age is equal to 3 add 3' or, 'Join the line if your age is half of 14'.

In the morning registration period, discuss with children how Numicon Shapes can be used to illustrate number problems that crop up every day, such as how many children are present and how many are absent. Try organizing a self-registration system for each child to place a tag on to show they are present. By placing the tags in Numicon Shape patterns, children can see at a glance how many of the class are present and how many are absent. By using Numicon Shapes and Numicon Shape patterns to illustrate these problems, all children can be involved.

Over time, this idea can be extended to include, for example, a birthday chart showing how many children are 6 years old and how many are 7 years old. In the photocopy masters section of the *Number, Pattern and Calculating 2 Teaching Resource Handbook*, you will find birthday charts for 'We are 6 years old' and 'We are 7 years old' (photocopy master 10a and 10b).

The routine of discussing the timetable each morning also provides a meaningful context for children to make predictions and develop mathematical language for describing temporal relationships (e.g. next, before, after, in between, nearly) that are also used to describe the order of numbers in relation to each other. Using a visual timetable enables all children to engage with this discussion.

Using daily opportunities and holding 'morning maths meetings' helps to ensure that children experience the breadth of the maths curriculum for each year and that they do not see mathematics as something that only happens in their mathematics lessons.



What might the use of Numicon look like in my classroom? Supporting communicating with a number-rich environment.

Numicon classrooms should be rich in number and visually indicate that mathematics is an important part of the children's everyday learning experience. Numicon images and numerals can be incorporated into labels and displays in many areas of the classroom, inviting children's attention. There will be number lines on display at children's eye level, including the Numicon Display Number Line and a Numicon 0–100 cm Number Line. In a corridor in the school, there may also be the Numicon 0–1001 Number Line for the children to engage with (available as part of the starter apparatus pack for Number, Pattern and Calculating 3 and 4).

At different times during the year, take the opportunity to put up mathematics displays that celebrate children's work: displays around a particular aspect of mathematics such as pattern, or size. These may include pictures, books and interesting objects, as well as children's work. It is useful to set up a mathematics table with an interactive display, where children can freely explore Numicon Shapes and number rods to practise activities that they have first met in focused teaching sessions.

All the classroom resources will be organized systematically, with storage trays numbered and stored in a logical order. Mathematics equipment will be easily accessible to children, so they can quickly find equipment when they need it – although it may also be set out on children's tables for them to use on specific activities.

This mathematically-rich environment itself provides children with valuable learning opportunities. Nearly all of us are acutely visually aware, and children are no exception. You will often notice them looking thoughtfully at number lines or other imagery in the classroom as they think (communicate with themselves). Throughout the day, and particularly when teaching the Numicon activities and solving mathematical problems, you will find children referring to the imagery and displays.

What could using Numicon feel like for children in my class?

By illustrating number ideas with the Numicon imagery in your teaching, numbers become real for children as they handle Numicon Shapes and number rods and use a variety of number lines with confidence.

Children also become aware that mathematics is something useful that they do at many points throughout the day. As you engage them in solving mathematical problems, they learn when to use the mathematics they are learning. Therefore, they do not see mathematics as something they do only in their daily mathematics lessons.

Children also feel supported by the Numicon imagery on display; you are likely to notice them glancing at images to check an idea they are explaining. At other times, you will notice them simply looking thoughtfully at imagery – they are likely to be noticing relationships and making connections as they assimilate new ideas.

Children look forward to working with Numicon materials and imagery and engage with the activities as the open-ended

nature of the resources invites them to experiment. While exploring an activity, children will move the Numicon Shapes and number rods around, use number lines and self-correct as they seek solutions. These opportunities to self-correct support their confidence.

Working in pairs within groups also supports children's confidence by providing ample opportunity for them to share ideas and work things out together. When they are working with a partner, many children appear confident, setting each other challenges and exploring ideas more deeply than they do when they are working alone.

Children take their lead for listening to each other and sharing ideas from the ways in which the adults in the classroom converse with them and each other. They will need help with taking turns, showing respect for each other's ideas, listening to one another without interrupting, with ways of phrasing questions and expressing ideas. Over time, you will find that children become confident in sharing their thinking.

One benefit of this is that, with Numicon, children know that there is nothing wrong with challenge: it is normal to get 'stuck'. What is important is that they continue to communicate mathematically in the face of the challenge. Children come to relish challenge because they feel able to persist in the face of it. They feel a sense of achievement when a challenge is overcome and then are excited about and ready for the next one. Part of this confidence comes from the ways in which their understanding is built cumulatively by following the suggested progression of teaching activities in the Teaching Resource Handbook. Teaching activities are introduced carefully, so that the challenges children meet are suitably challenging.

Children also feel confident when tackling new ideas because they can use Numicon Shapes, number rods and other imagery to illustrate problems in various ways. Their confidence will be further encouraged as you discuss their ideas with them, helping them to become increasingly aware of what they know and are learning, as they explain their thinking. This discussion supports children's monitoring of their own learning.

Is there a risk of children becoming over-dependent on the imagery?

Numicon imagery offers children structured illustrations of numbers to encourage understanding of number relationships in ways not provided by written numerals. To support the development of children's mental imagery, it is important to encourage children to visualize the Numicon Shapes, number rods and number lines. This can be supported by activities with the Feely Bag, where children identify Numicon Shapes and number rods by touch, and by activities where they group objects into patterns of all the Numicon Shapes without counting. Such activities are described in the Getting Started activities within the Securing Foundations section. As you work with children, remind them to refer to the Numicon images displayed around the classroom.

You will find that children like to work efficiently and quickly and, once they are secure with an idea, they tend to move on to working in their heads, using their own mental imagery. However, when children are consolidating an idea or meeting a new one, they will often use the physical materials and imagery until they are well on the way to understanding.

Thus, there is no point at which imagery becomes redundant; meeting new mathematical ideas is on-going and illustrating thinking with structured imagery gives children of all ages and abilities a practical, accessible way to investigate mathematical problems.

How much time should I plan to spend on mathematics teaching?

The time spent teaching mathematics can vary as there are many opportunities to discuss mathematics across the school day. In addition to a daily mathematics lesson lasting up to an hour and a 'morning maths meeting' lasting about fifteen minutes, there will be opportunities for developing language of comparing, position, time and shape that will inevitably crop up.

There will also be many opportunities for calculating – such as: 'Have we got enough pencils for everyone in the group to have one?', 'Is there enough room for three chairs in this space?' or 'How long is it until play time?' Try to also refer to current events in the school, e.g. 'How much money will we raise if everyone brings 50p to our charity day?' Solving these problems helps children realize that doing mathematics is useful in all sorts of situations and gives them opportunities to use the mathematics they are learning as they work out what they need to use to solve the problem.

What format might Numicon mathematics lessons take?

Look at the relevant pages of the Teaching Resource Handbook for the activity group you are working from. The first part of the mathematics lesson is a whole-class introduction around this. This is followed by a longer session of group work during which children are either working independently or being taught in a focused teaching group. Finally, the class comes together for a concluding session.

During the introductory session, you are likely to use Numicon Shapes, number rods, number lines and other imagery in your communicating with children. You may also use the *Numicon Software for the Interactive Whiteboard.*

Children may join in this whole-class part of the lesson in lots of different ways: participating in a class conversation; talking with a partner or discussing in a small group; jotting on an individual whiteboard; or holding up Numicon Shapes or number rods to show their ideas.

In the second part of the lesson, children will be arranged in groups working on, for example:

- a teacher-led focused teaching activity
- a teaching assistant-led activity
- an independent practice activity or investigation, or further work on ideas introduced at the start of the lesson.

Groups of children will be exploring ideas from the same activity group but are unlikely to be working on an identical activity. The various groups may be using different apparatus, e.g. for a lesson on adding, some groups may be using Numicon Shapes, others may be using number rods, others may be writing and drawing their own adding stories.

Over the course of a week, the different groups of children will rotate around the various group-work activities, so that all children receive focus teaching and explore the ideas using different imagery.

In the final part of the lesson, it is particularly important to encourage all children to reflect on their learning by asking questions of children working at different levels, depending on what you have noticed them doing and saying during the lesson. You may decide to ask the different groups to explain to the rest of the class what they have been doing and what they have noticed. You may also have points you want to draw to children's attention. You could suggest what might happen in the next lesson and anything children could think about before then. To help children reflect on their lesson, you may ask them to think quietly for a few moments about what they have been doing and guide their reflection with questions, such as:

- Is there anything new you have learned?
- Is there anything you feel particularly pleased about?
- Is there anything you particularly enjoyed?
- Is there something that you found difficult?
- Is there something that you are still puzzling about?
- Is there something you would like to do again?

Questions such as this will encourage children to reflect and become consciously aware of what they are learning.

How will I manage the class for the mathematics lessons?

There are several important factors to consider for organizing and managing successful group work.

Firstly, to engage children, it is important to plan differentiated activities that will provide appropriate levels of challenge for all children. Therefore, during the week, the group-work activities will need to be adjusted for different groups.

Secondly, as children may be working on activities that have been appropriately adjusted to make them suitable, it is important that they are comfortable with how to do the activity and what is expected of them.

Thirdly, the order in which the activities are introduced to different groups has an impact on how quickly children progress with the activities.

At the beginning of the week, the class may be taking in a lot of information, so give consideration to how you will introduce the activities as this will impact on how quickly children can access and begin to make progress through the work. You will find ways that work well for you, but the following guidance may be of help:

- Explain the simplest independent work first.
- The first time you introduce a challenging practice activity, allocate it to a group of children who are able to follow instructions well.
- Groups that might need more support should begin the week with a focus activity; the adult working with the group can explain the activity, removing the need to spend time explaining it to the whole class.

Once the activities have been introduced, children will then go off to work in their groups. You may wish to focus on the one or two groups that need the most support at this point. If you have a teaching assistant, you might want them to work closely with the group who need the greatest assistance. Each class, and each lesson, will be different.

During this part of the lesson, it is quite likely that a child or children will put forward an idea that is worth everybody considering. In which case, you might choose either to invite all children to take a moment to reflect on the idea, or to make a note to discuss this in the final session when the class comes back together.

What writing or drawing might children do in mathematics if the activities are mainly practical?

Writing and drawing are aspects of children's communicating and will take many forms. Bear in mind that this is their communicating, so it may be idiosyncratic and include: drawing a number story with pictures and Numicon Shapes, drawing around Numicon Shapes, drawing repeating patterns, drawing objects arranged in Numicon Shape patterns, using number lines, making adding and subtracting number sentences, e.g. 10 = 3 + 7 or 10 − 3 = 7.

There are opportunities signalled in the activities for children to communicate on paper wherever this serves a useful purpose. Amongst the photocopy masters there are some activity sheets, but in general it is recommended that children write and draw in their mathematics exercise books. This provides a useful bank of evidence that shows how children's communicating is developing over time.

Children's Explorer Progress Books (see page 40 for more information) will also provide an extremely useful source of evidence for monitoring how children are progressing throughout the year.

Giving children the choice of how to communicate their ideas can provide useful insights into how they are approaching problems, whether they are working systematically and how they are using conventional notation.

What about organizing resources?

It is worth considering that the way in which the resources are organized, used and presented to children as part of the daily classroom set-up gives a strong message about how children are expected to use the resources.

When preparing the classroom for mathematics lessons, set out the equipment on the tables where groups of children are to work. Refer to the '**Have ready**' section in the activities you plan to teach. The photocopy masters can be photocopied from the *Number, Pattern and Calculating 2 Teaching Resource Handbook* or printed from the Numicon Planning and Assessment Support.

If you have planned for all the groups to have a turn at each of the activities over a week, it can be helpful to sort the equipment needed for each activity into a tray or basket and store these so the equipment is ready for use each day. Encourage children to leave the resources as they found them, ready for the next group to use. Children can help themselves to any further resources they need from the class mathematics resource area. As children become more used to working with the Numicon apparatus, they can begin to collect the essential equipment they will need for an activity by reading a list (illustrated if necessary with pictures, symbols or the drawer number), and think for themselves about any further equipment they may decide to use.

Other ideas for organizing resources include:

- having smaller baskets of equipment for children working individually or in pairs
- printing or photocopying sets of numeral cards onto different coloured card so sets can be kept together easily
- using sorting trays with separate compartments to allow a group of children to access numeral cards to save space and keep the cards organized
- storing Numicon 0–100 cm Number Lines by hanging them from a hook (punch a hole in the end).

What about grouping children?

Children work well in pairs within larger groups of four or six. Working with a partner supports their confidence, giving them plenty of opportunities to discuss their work and solve problems together.

Schools vary in their criteria for grouping children. When grouping children of six and seven years old, it is important to bear in mind that they will be at very different levels of development and will therefore require different levels of challenge. Some schools find an advantage in having mixed-ability groups comprising pairs of children working at slightly different levels; others group together children who are working at a similar level.

Some children will work faster than others and have more developed ideas. It is important to make sure that they have opportunities to work with other children of similar ability.

Whatever the policy is in your school, bear in mind that it is important to vary groups from time to time and to ensure that children do not always work with the same partner.

As you follow the suggested teaching programme for the first few weeks of term, you will discover which children work well together and their levels of understanding.

How do I prepare for teaching mathematics lessons using Numicon?

Understanding the mathematics yourself

Before teaching an activity group, read the relevant sections from the 'Key mathematical ideas' (page 42) to prepare for your teaching. If you want to do more research on an area of mathematics, you may wish to consult other sources, for example Derek Haylock's *Mathematics Explained for Primary Teachers* (3rd ed, 2006).

Next, consider what generalizing there is in the activity group. For example, 'Is this where children could notice that it doesn't matter which way round you add two numbers, the total will always be the same?' Or, 'Is this where children notice that if they add a multiple of ten to a number, the ones figure doesn't change?'

The children will be new to this, so another way to work on generalizations is to ask, 'What *patterns* in the work children are doing will they have to notice in order to progress?'

When children notice things, be prepared to keep asking: 'Will that always work?', 'What if those numbers were different?', 'Would that work with fractions?', 'Will that ever work?', '*When* does that work?' and 'What never works?'

Appreciating the contexts

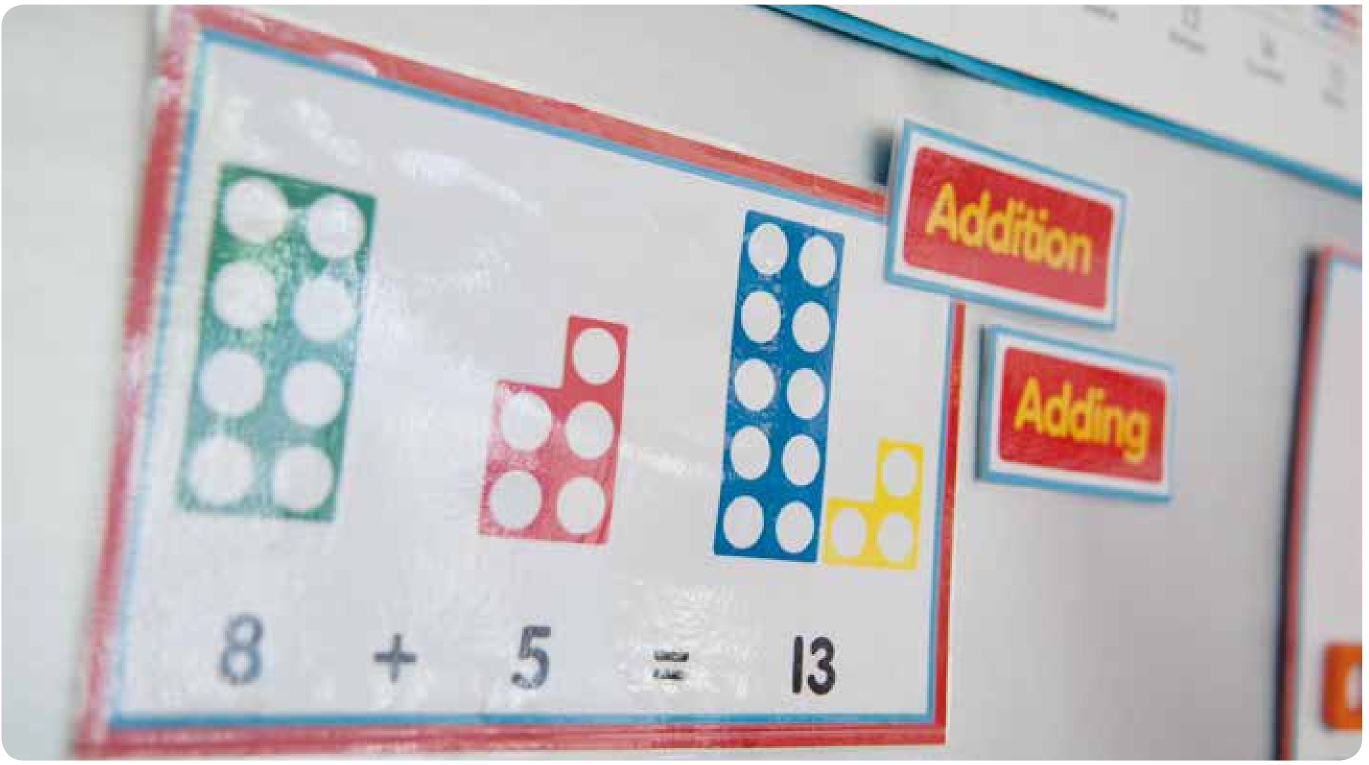
The educational context on the introductory page for each activity group will help you to see how the ideas involved fit into the continuum of children's learning about Number, Pattern and Calculating.

After reading the educational context of an activity group, consider when this learning may be useful. Think about the kinds of contexts offered in the activity group; is this mathematics useful in particular kinds of real-world situations, or will it help me do some other mathematics? It can be helpful to think up one or two contexts of your own, so that it is clear what the point is of doing this mathematics.

Children do not just need to know how to do this mathematics, they need to know *when* to do it, as well. How can you help them spot when this general mathematics applies to a particular situation?

Understanding the illustrating and communicating involved

Since so much mathematics involves working with generalizations, children will need illustrations to help them understand what they are talking about. Consider what illustrations are available to you and the children to support work on a particular activity group. Think about the ways the available illustrations might help children 'to see the general in the particular'. For instance, if you want children to generalize



that 'it doesn't matter which way round you add two numbers, the total will always be the same', then using visual Numicon Shapes or number rods as illustrations is very immediate.

Study the illustrations offered in an activity group. Consider if there are any further illustrations that could be used to emphasize key points. Doing mathematics involves exploring the relationships in a situation. Give thought to which illustrations will help children explore these relationships best.

Think about which symbols and words are key to successful communicating in this area and whether any of these are new. Activity groups suggest key terms; the idea is that, through doing the activities, the children will learn how such terms are used. Encourage children to use them in their conversations and let them notice how you use them.

Selecting and adapting activities

Read all of the activities in an activity group and identify what each activity contributes to the overall work. Then try the activities.

You know the children of your class and the materials available to you. You will be best placed to select which activities are most appropriate and adapt them creatively to suit the needs of the individual children.

Some activities might be revision for your children, others might involve ideas, illustrations or techniques that are completely new. Some activities will be invaluable practice. You might think an activity will be too easy/difficult for some children, so consider how you might make it more/less challenging. Be flexible; adapt what is available for your children in light of what they can already do.

It is also worth considering the fact that facing challenge in every subject is normal; when children get 'stuck', they should be encouraged to communicate. Make sure they have available all the actions, imagery and conversation they might need to communicate effectively about the level of challenge they are facing.

How do I plan in the long- and medium-term using the Number, Pattern and Calculating 2 teaching programme?

The plan–teach–review cycle applies to Numicon, just as it applies to all effective mathematics teaching. There are, however, four important features of Numicon that support this cycle.

Firstly, the Numicon teaching programme (the suggested order of teaching the activity groups) is structured progressively. This chart can be found in the long- and medium-term planning section of the *Number, Pattern and Calculating 2 Teaching Resource Handbook* (also available as an editable version in the Numicon Planning and Assessment Support).

The second feature is that there are practice and discussion activities within each activity group, some for individual work and others for paired work.

Thirdly, assessment can be more accurate through children's practical work with Numicon materials and imagery and communicating their ideas (through talk and on paper). These assessments will, in turn, help with planning.

Finally, 'using and applying' does not need to be planned separately. This is partly because all the activity groups involve problems that need to be solved, but also because the cumulative nature of the teaching programme means that children are using their earlier ideas every time they face a new one.

The teaching programmes throughout Number, Pattern and Calculating are arranged into three strands: Pattern and Algebra, Numbers and the Number System and Calculating. Within each strand is a sequence of activity groups, though the strands are interrelated and what children learn in one strand supports their learning in another.

The long-term plan from the *Number, Pattern and Calculating 2 Teaching Resource Handbook* shows the recommended order for teaching the activity groups. This plan has been carefully designed to scaffold children's understanding so that they are able to meet the challenges of each new idea, e.g. children would not be expected to learn how to add whole tens until they have an understanding of place value and recall of adding facts of numbers 1–10.

The medium-term planning guide from the *Number, Pattern and Calculating 2 Teaching Resource Handbook*, gives the expected coverage over the course of the year and also lists the activities and the learning opportunities for each group. The medium-term plan also lists the milestones that children need to be secure in as they progress through the teaching programme. These milestone statements should be used to assess children's progress throughout the year.

You may decide to follow the long- and medium-term plans as they stand. You may also find that you need to split some of the larger activity groups and return to them later.

There are summary charts showing the title and learning opportunities for each activity group on the Numicon Planning and Assessment Support. You may find these useful for incorporating Numicon activities into your existing mathematics plans should you decide not to follow the Numicon long-term plan for teaching the activity groups.

The section on 'Using the activity groups' on pages 34–35 of this Implementation Guide (also included in the Teaching Resource Handbook) highlights the key parts of each activity group.

Each of the activity groups begins with a 'low-threshold' focus activity, designed deliberately to support confidence and ensure that all children are included. The remaining focus activities are designed to help children progressively develop their ideas around the theme of the activity group. You will notice that there are opportunities for reasoning about numbers throughout focus activities through challenging questions.

The focus activities are designed for whole-class or group teaching. Some may be taught quite quickly to the whole class as an introduction to be explored later with a focus group.

Ensure that activities are differentiated where necessary so that all children who should be working independently can do so. Include activities that allow children to become more confident and allow them to be able to work more speedily by practising and celebrating what they are able to do. As you decide which activity to allocate to different groups of children, remember to check that there is scope for children to take the activity further. You can increase the challenge through your questions with specific groups and by asking

challenging questions when the class comes together for the final part of the lesson. Your questions will depend on what you have noticed the children doing and saying during the lesson.

How long should I allow for teaching each activity group?

The Numicon teaching activities for Number, Pattern and Calculating 2 primarily address the number and algebra areas of the curriculum. There are 30 activity groups within the *Number, Pattern and Calculating 2 Teaching Resource Handbook*.

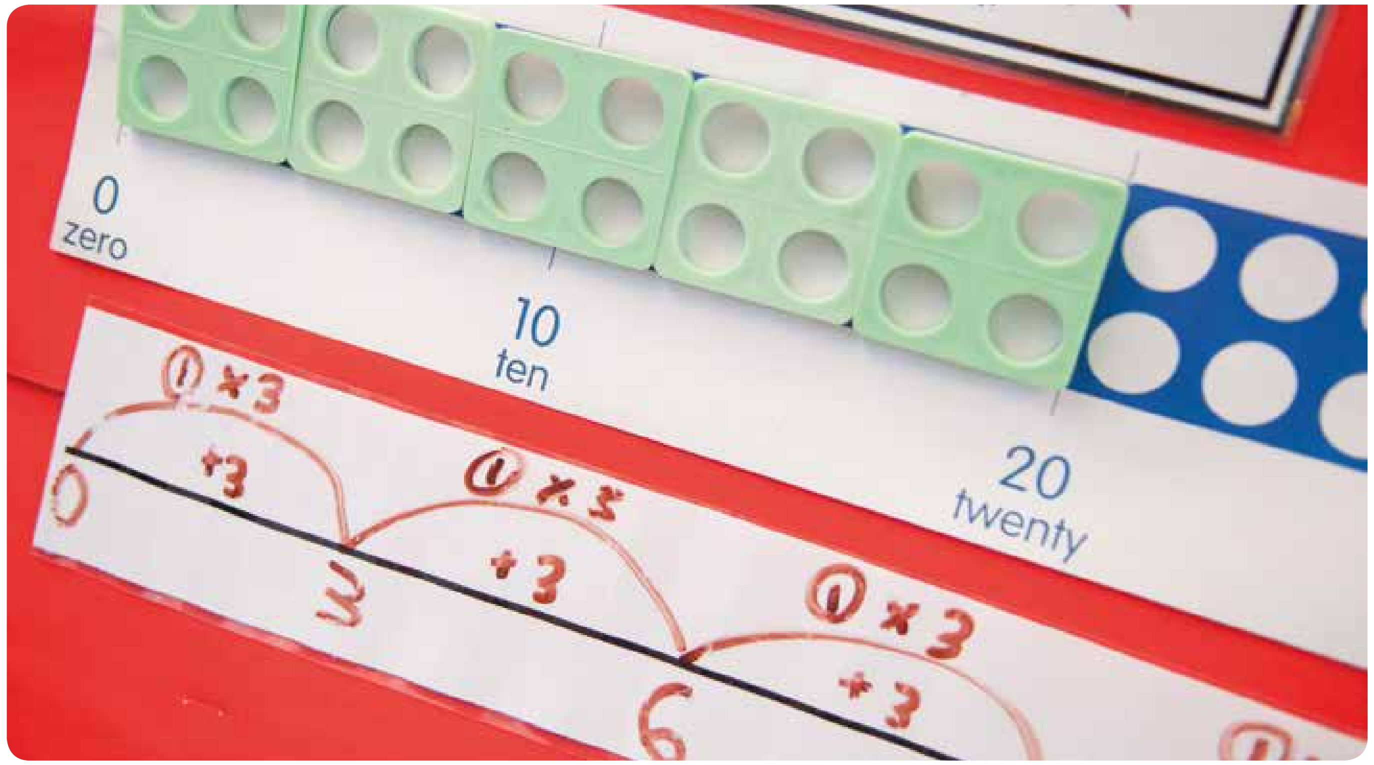
For children who are new to Numicon, you will need to allow plenty of time to cover the activities in the Getting Started activity group as it is essential for children's success with Numicon that they are able to make connections between Numicon Shapes and Patterns, number rods, number names and numerals.

Look through the long- and medium-term plan for Number, Pattern and Calculating 2 to assess how much time you think will need to be given to each activity group, considering the requirements of the children in your class.

Careful consideration will need to be given about how much of each activity group is essential for the children in your class to complete before moving on. It will also be crucial for you to leave sufficient time for the activity groups at the end of the Numicon teaching programme for Number, Pattern and Calculating 2 so that children don't miss out on content that happens later in the school year.

Within each activity group, a range of focus activities have been provided. The Numicon teaching progression provides flexibility for teachers to exercise their professional judgment so that, if children are confident and secure with the ideas in the activity group, you can quickly move them through the activity group, whilst retaining the confidence about where to go next. As you teach the activity groups, you will find that some of the children will move very quickly and you will be able to combine two or sometimes even three focus activities from a group in one teaching session.

When deciding how many of which activities to utilize from an activity group, refer to the relevant milestone statements from the medium-term planning to guide you in your selection of material for your children. Milestones pick out those aspects of mathematics that are absolutely essential for children's progress, and so specify what they cannot afford to miss.



As you plan for your 'morning maths meeting', build in practice of counting and number facts from earlier activity groups to help children develop fluent recall. There are some activity groups, for example, those where lots of practice is needed to develop fluency with adding and subtracting facts of numbers to 10, which children will perhaps need to revisit often over a longer period alongside work on other activity groups either from Number, Pattern and Calculating 2 or from Geometry, Measurement and Statistics 2. The educational context of these activity groups will highlight if extra time is likely to be needed for new and difficult ideas. Ideas can be kept bubbling by revisiting whole-class practice suggestions at different times during the day and, in particular, during morning maths meetings.

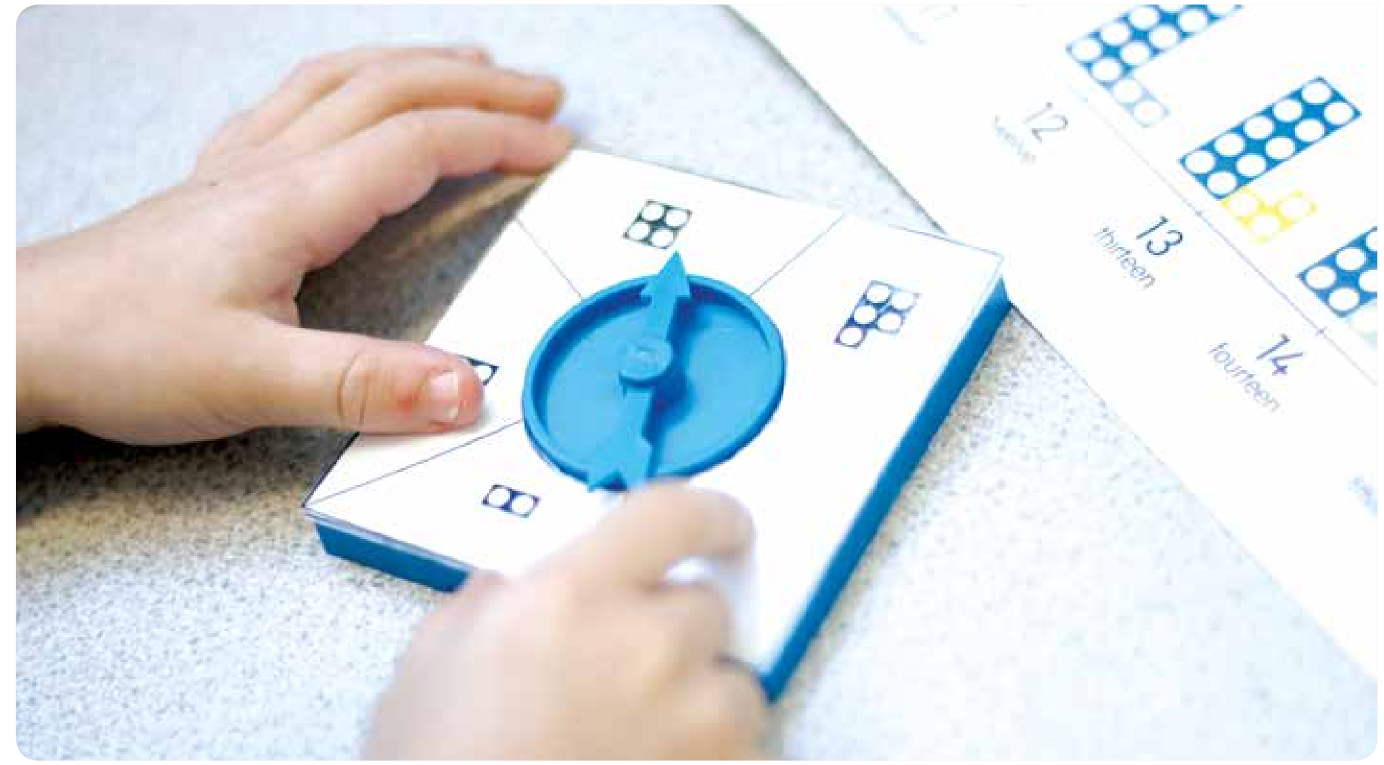
What about differentiating activities?

It is important for children of all abilities that activities are differentiated with appropriate levels of challenge. The learning opportunities and the educational context of the activity group give you an overview of the ideas children will be meeting, and the previous learning being built on.

If your earlier assessments have told you that some children are not yet ready for the level of challenge in the activity group, look back through earlier activities in the same strand to find appropriate activities. For some children, it may be appropriate to differentiate by adjusting the number range within the activities. For some groups of children, particularly in mixed-age classes, you may need to look back to earlier activities from the same strand in the *Number, Pattern and Calculating 2 Teaching Resource Handbook*. Alternatively, you may identify suitable activities from the *Number, Pattern and Calculating 1 Teaching Resource Handbook*.

Each activity group starts with a 'low-threshold' activity which is designed to be accessible to all children (although in a mixed-age class you will need to modify the work for the younger children and assess how they respond). You may decide that other children are ready to go straight to the more challenging activities later in the activity group and you will find the open-ended nature of the activities and the emphasis on mathematical thinking means that there is always room for children to take activities further.

For the highest-achieving children, you may decide to increase the challenge through planning specific questions that extend the reach of the activity.



How can I support children to develop fluency?

At the end of each activity group is a list of suggestions for whole-class and independent practice and discussion activities to help children develop fluent understanding of the ideas they are meeting in the activity group. You can select from these to give children appropriately differentiated opportunities that will help them to develop fluency and confidence with the maths they are learning.

Explore More Copymasters provide further opportunities for children to practise and discuss at home the ideas they have been working on at school.

The 'morning maths meeting' also provides an excellent opportunity for children to practise and discuss ideas.

Maintaining children's fluency

Children's responses to mental mathematics questions, word problems, and their ability to make up their own problems in 'morning maths meetings' or in practice sessions will inform you whether they are maintaining fluency with past learning.

You can also keep track of what children have remembered by choosing an activity from the practice suggestions of a completed activity group, varying the context of the problem and presenting it to children without preparing them in advance. Notice what they seem to have remembered and what they have not remembered. Plan accordingly when you reach the next activity group in that section.

In the whole-class and group-focus teaching sessions, choose some questions and activities from previous activity groups that will help to keep children's past learning 'simmering'.

Using the activity groups

The first page of each activity group is clearly coloured according to the **strand** it appears in (Getting Started – light blue, Pattern and Algebra – red, Numbers and the Number System – yellow, Calculating – dark blue). The title and the numbering of the activity group allow you to easily identify the content of the activity group and how far through the strand you are.

The **key mathematical ideas** clearly highlight the important ideas children will be meeting within each activity group.

The **assessment opportunities** signal key information to 'look and listen for' that indicate how much of the focus activities children have understood.

The **educational context** gives a clear outline of the content covered in the activity group, for example: how it builds on children's prior learning; how it connects with other activity groups; the foundation it establishes for children's future learning.

All activity groups have been extensively trialled in the classroom, so the **learning opportunities** come from real classroom experiences. They are designed to help children develop their understanding of the key ideas of an activity group.

Key mathematical ideas Counting, Pattern, Order, Place value, Equivalence, Mathematical thinking and reasoning **Numbers and the Number System**

Comparing and ordering numbers to 100 4

Educational context

This group of activities focuses on reasoning to make comparisons between number values which can then be used in the context of measures. Children have opportunities to continue to use the symbols for greater than (>) and less than (<) to record comparisons. As children compare and order higher numbers they will need to have a clear understanding of place value, i.e. that the place of a digit tells us its value. Comparing and ordering Numicon Shapes and number rods makes visible the important regularity in the order of numbers, which is a crucial step towards understanding the system of whole numbers. Listen for any children who do not speak clearly and run 'than' into the previous word, e.g. saying 'biggeran' instead of 'bigger than', as 'than' is a key word used in a comparison.

Learning opportunities

- To recognize when it is helpful to use the order of numbers to organize or find things.
- To use the '<' and '>' symbols when comparing Numicon Shapes, number rods and numerals.
- To compare and order numbers to 100.

Words and terms for use in conversation

tens, ones, more, less, between, nearly, next, before, after, forwards, backwards, larger than, greater than, bigger than, smaller than, more than, less than, fewer than, higher, lower, 'I know this, so I know that'

Assessment opportunities

Look and listen for children who:

- Use the words and terms for use in conversation effectively in discussion.
- Enunciate the word 'than' clearly to say, e.g. 'larger than' and not 'largeran'.
- Are well-organized and recognize order.
- Describe comparisons and infer, e.g. 'I know this, so I know that.'
- Use the '<' and '>' symbols to record comparisons.
- Explain that numbers with more tens are larger than numbers with fewer tens.
- Use the word 'between' effectively.
- Make size comparisons between numbers in the range 0–100.
- Can put a list of up to seven numbers from the range 0–100 in order.
- Spell number words at a level consistent with their spelling knowledge.

Explorer Progress Book 2a, pp. 20–23

After completing work on this activity group, give small focus groups of children their Explorer Progress Books and ask them to work through the challenges on the pages. As children complete the pages, assess what progress they are making with the central ideas from the activity group. Refer to the assessment opportunities for assistance.

Children will also have the opportunity to complete their Learning Log (pp. 22–23) where they can reflect on the mathematics they have done so far.

Explore More Copymaster 12: Biggest Number

After completing work on Activity 3, give children Explore More Copymaster 12: Biggest Number to take home.

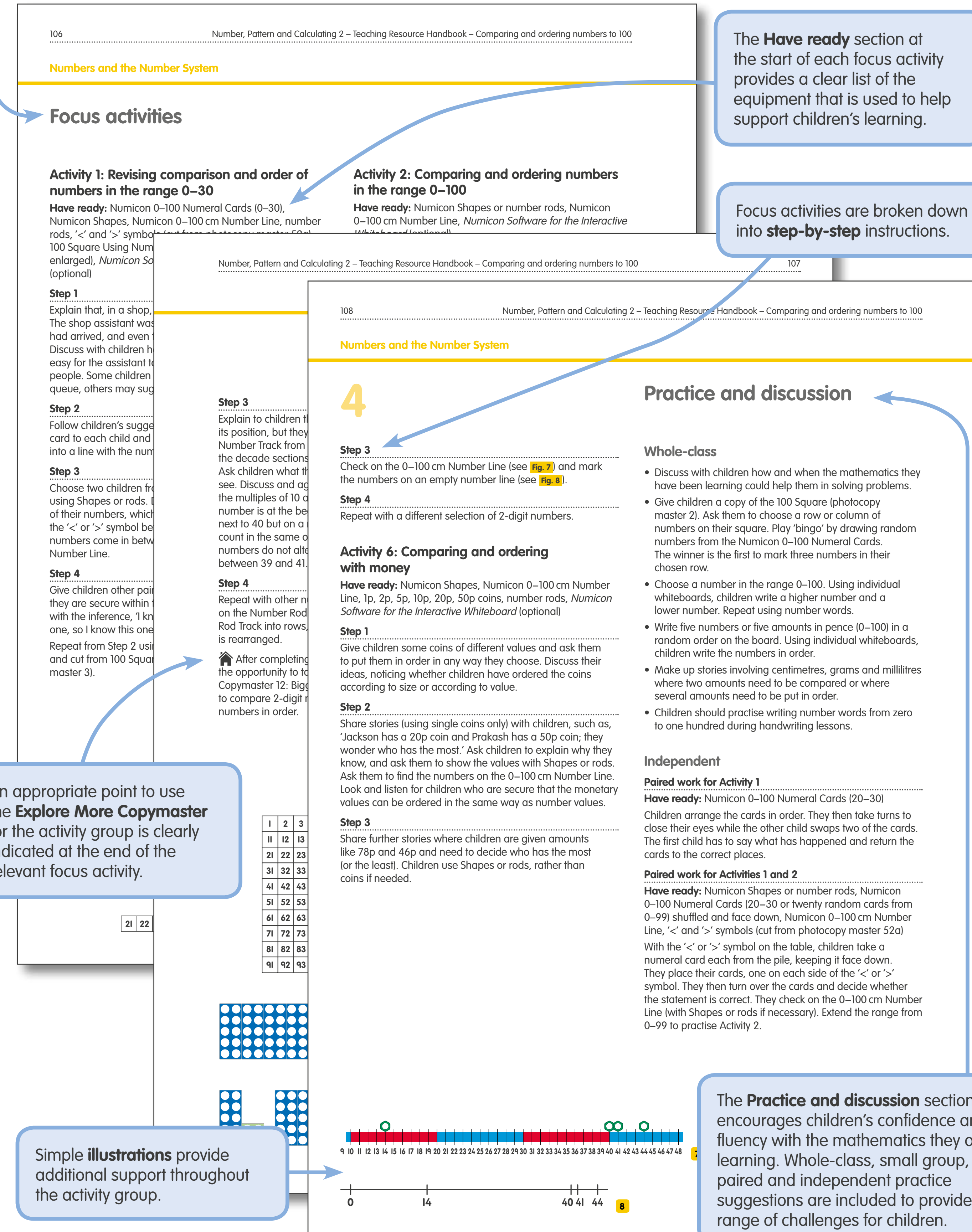
Explore More Copymasters provide an opportunity for children to practise the mathematics from the activity group outside the classroom through fun, engaging activities.

Clear links are made to the **Explorer Progress Books**. These books provide an invaluable chance to see children's thinking, monitor their progress and assess how much of the activity group they have understood.

Each activity group includes several **focus activities**, each clearly titled to show the specific learning points addressed by the activity. The first focus activity is a 'low threshold' activity allowing all children to engage with the work. Focus activities then build progressively to a 'high ceiling' that provides challenge and allows for differentiation within the same activity group.

The **Have ready** section at the start of each focus activity provides a clear list of the equipment that is used to help support children's learning.

Focus activities are broken down into **step-by-step** instructions.



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Numbers and the Number System

Focus activities

Activity 1: Revising comparison and order of numbers in the range 0–30

Have ready: Numicon 0–100 Numeral Cards (0–30), Numicon Shapes, Numicon 0–100 cm Number Line, number rods, '<' and '>' symbo… 100 Square Using Num… enlarged), Numicon So… (optional)

Step 1 Explain that, in a shop, … The shop assistant was … had arrived, and even … Discuss with children h… easy for the assistant t… people. Some children … queue, others may sug…

Step 2 Follow children's sugge… card to each child and … into a line with the num…

Step 3 Choose two children fr… using Shapes or rods. … of their numbers, which … the '<' or '>' symbol be… numbers come in betw… Number Line.

Step 4 Give children other pai… they are secure within … with the inference, 'I kn… one, so I know this one…

Repeat from Step 2 usi… and cut from 100 Squar… master 3).

Activity 2: Comparing and ordering numbers in the range 0–100

Have ready: Numicon Shapes or number rods, Numicon 0–100 cm Number Line, *Numicon Software for the Interactive Whiteboard (optional)*

Number, Pattern and Calculating 2 – Teaching Resource Handbook – Comparing and ordering numbers to 100 107

Step 3 Explain to children t… its position, but they … Number Track from … the decade sections … Ask children what t… see. Discuss and ag… the multiples of 10 a… number is at the be… next to 40 but on a … count in the same o… numbers do not alte… between 39 and 41.

Step 4 Repeat with other n… on the Number Rod… Rod Track into rows… is rearranged.

After completing … the opportunity to t… Copymaster 12: Big… to compare 2-digit … numbers in order.

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Numbers and the Number System

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Step 3 Check on the 0–100 cm Number Line (see Fig. 7) and mark the numbers on an empty number line (see Fig. 8).

Step 4 Repeat with a different selection of 2-digit numbers.

Activity 6: Comparing and ordering with money

Have ready: Numicon Shapes, Numicon 0–100 cm Number Line, 1p, 2p, 5p, 10p, 20p, 50p coins, number rods, *Numicon Software for the Interactive Whiteboard* (optional)

Step 1 Give children some coins of different values and ask them to put them in order in any way they choose. Discuss their ideas, noticing whether children have ordered the coins according to size or according to value.

Step 2 Share stories (using single coins only) with children, such as, 'Jackson has a 20p coin and Prakash has a 50p coin; they wonder who has the most.' Ask children to explain why they know, and ask them to show the values with Shapes or rods. Ask them to find the numbers on the 0–100 cm Number Line. Look and listen for children who are secure that the monetary values can be ordered in the same way as number values.

Step 3 Share further stories where children are given amounts like 78p and 46p and need to decide who has the most (or the least). Children use Shapes or rods, rather than coins if needed.

Practice and discussion

Whole-class

- Discuss with children how and when the mathematics they have been learning could help them in solving problems.
- Give children a copy of the 100 Square (photocopy master 2). Ask them to choose a row or column of numbers on their square. Play 'bingo' by drawing random numbers from the Numicon 0–100 Numeral Cards. The winner is the first to mark three numbers in their chosen row.
- Choose a number in the range 0–100. Using individual whiteboards, children write a higher number and a lower number. Repeat using number words.
- Write five numbers or five amounts in pence (0–100) in a random order on the board. Using individual whiteboards, children write the numbers in order.
- Make up stories involving centimetres, grams and millilitres where two amounts need to be compared or where several amounts need to be put in order.
- Children should practise writing number words from zero to one hundred during handwriting lessons.

Independent

Paired work for Activity 1

Have ready: Numicon 0–100 Numeral Cards (20–30)

Children arrange the cards in order. They then take turns to close their eyes while the other child swaps two of the cards. The first child has to say what has happened and return the cards to the correct places.

Paired work for Activities 1 and 2

Have ready: Numicon Shapes or number rods, Numicon 0–100 Numeral Cards (20–30 or twenty random cards from 0–99) shuffled and face down, Numicon 0–100 cm Number Line, '<' and '>' symbols (cut from photocopy master 52a)

With the '<' or '>' symbol on the table, children take a numeral card each from the pile, keeping it face down. They place their cards, one on each side of the '<' or '>' symbol. They then turn over the cards and decide whether the statement is correct. They check on the 0–100 cm Number Line (with Shapes or rods if necessary). Extend the range from 0–99 to practise Activity 2.

An appropriate point to use the **Explore More Copymaster** for the activity group is clearly indicated at the end of the relevant focus activity.

Simple **illustrations** provide additional support throughout the activity group.

The **Practice and discussion** section encourages children's confidence and fluency with the mathematics they are learning. Whole-class, small group, paired and independent practice suggestions are included to provide a range of challenges for children.

Planning and assessment cycle

Here is a guide to show how planning can be informed by your assessments of children's understanding.

1. Choose an activity group	Before embarking on the activity group, review the educational context, consider your initial assessments of children and decide whether there is a need to revise any ideas children have met earlier.
2. Choose a focus activity	If this is the first lesson using the activity group, start with an early 'low threshold' activity to include all children. Before the lesson, check the learning and assessment opportunities, brief teaching assistants and prepare resources.
3. Choose the practice activities	**Independent groups:** Refer to your assessment notes and allocate suitable practice activities (found at the end of the activity group). **Focus teaching groups:** Refer to your assessment notes and the learning and assessment opportunities from the activity group and allocate a focus activity.
4. Plenary session (normally during and at the end of lessons)	Think about the important ideas that children will have met in the lesson, particularly any generalizations that you want children to have made. Plan questions to prompt discussion and ask questions that encourage children to reflect on ideas they may have learned. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
5. After the lesson	Reflect on how children have responded in the lesson and note any significant steps on your assessment records. Use what you notice to determine the plan for the next lesson. Suggestions are given for whole-class practice that will help children develop the ideas they have learned in the lesson. At some point after children have completed work on the activity group, ask them to complete the relevant pages of their Explorer Progress Book. This will allow you to assess how well they have retained the information they have been learning. It will also give you a chance to see how well they are able to apply this knowledge when faced with a 'non-routine' problem.

Creating short-term plans

	Warm-up	Main Teaching Focus	Focused Group Work with the Class Teacher or Teaching Assistant	Independent Work	Plenary
Activity number/title	Select activities from the **Whole-class practice** section. This could be from a previous activity group to review and refresh children's previous learning.	Select one of the focus activities from the activity group matched to the needs of the children. Place the **activity number/title** of the chosen focus activity in your short-term plan.	Decide whether to: • select the next **activity number/title** from the focus activities in the activity group. Place this in your short-term plan; or: • consolidate the activity covered in the main teaching focus.	Decide whether to: • choose activities from the **Independent practice** section for groups, pairs or individual children. Make notes on your plan or work from the Teaching Resource Handbook; or: • select a focus activity for groups to work on independently. Place the relevant **activity number/title** in your short-term plan.	Encourage children to have a reflective conversation to draw together what has been learned in the lesson. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
Learning opportunities	Place the selected learning opportunity(ies), from the chosen activity group summary in your short-term plan.				
Notes and Educational context	Decide whether to: • use the activity directly from your *Number, Pattern and Calculating 2 Teaching Resource Handbook*; or: • draw on the Teaching Resource Handbook to make your own notes for teaching the activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 2 Teaching Resource Handbook*; or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 2 Teaching Resource Handbook*; or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity. If working with a teaching assistant, you may want to select the relevant **Educational context**, from the chosen activity group.	Decide whether to: • use the practice or focus activity from your *Number, Pattern and Calculating 2 Teaching Resource Handbook*; or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	
Words and terms	Decide which **words and terms** you will use in **conversation**. Place these in your short-term plan.				
Resources	Prepare any resources you may need for the activity. Use the **have ready** section at the beginning of the focus and practice activities.				
Assessment opportunities	Select from the chosen activity group the **assessment opportunities** that you and the teaching assistant will be looking and listening for in the different parts of the lesson. Place these in your short-term plan. Remember to note whether children know when to use their understanding.				

How can I assess children's progress?

Assessing mathematics using Numicon involves making judgments about developments in children's mathematical communicating – both receptive and expressive.

As a result, you will need to know what the key developments are that you should look for. For this, you can check the assessment opportunities signalled in each activity group and consider how these achievements would show up in children's mathematical communicating. Specifically, you will need to look for developments in children's actions (what they do and notice), the imagery they use and respond to and their use of (and responses to) words and symbols in their conversation.

It is also important to notice children's fluency. For example, when is their communicating stilted, when is it punctuated by gaps and hesitations, and when does it flow consistently and well, suggesting a strong command of connections between well-established ideas?

Assessment should be as open as possible so that children can communicate as much as possible. It is through their mathematical communicating that you will gain a real insight into how children are thinking. This will enable you to make the most accurate assessment of their progress.

Specific challenges for the purposes of assessing are provided in the form of *Explorer Progress Books* (see page 40). Children cannot pass or fail at these assessment tasks, they simply respond in their individual ways. How they approach the tasks and their responses to them will inform you about their mathematical communicating and give you an opportunity to 'see' their thinking through the illustrating they use within the tasks. This level of insight into children's thinking will make it easier to gather meaningful and accurate assessment of where children are. Preparing for formal test situations is something different and is addressed on page 41.

Specific indications of children's progress

Each activity group lists several assessment opportunities that point to key achievements to look for during the work of that activity group. All of these achievements will be evident in children's actions, imagery and conversation as they progress.

Familiarize yourself with these before you begin teaching any activity group. They will help guide your interactions with children as they tackle the activities you give them. They will also indicate progress and provide useful information for grouping children and planning your teaching as you move on to the remaining activity groups.

Within each activity, there are also suggestions for what to 'look and listen for' as children are working on the activities. Focus on children's communicating and ask whether they know *how* to do the mathematics they are learning, and whether they know *when* to use it.

You will also find that how children use the Numicon Shapes and number rods gives you insight into their thinking.

A child using Numicon Shapes by trial and error with muddled explanation would suggest they do not yet understand the activity. Plan to revisit it, focusing carefully on the mathematical language and imagery you will use. It may be that the child did not understand what they had to do.

Children self-correcting, i.e. working by trial and improvement, suggests their understanding is developing as they try out different solutions. Give children time to experiment and practise the activity and encourage discussion about their ideas.

Children communicating clearly about what they have done (by talking, with apparatus or through writing it down), suggests solid understanding, so plan to move them on.

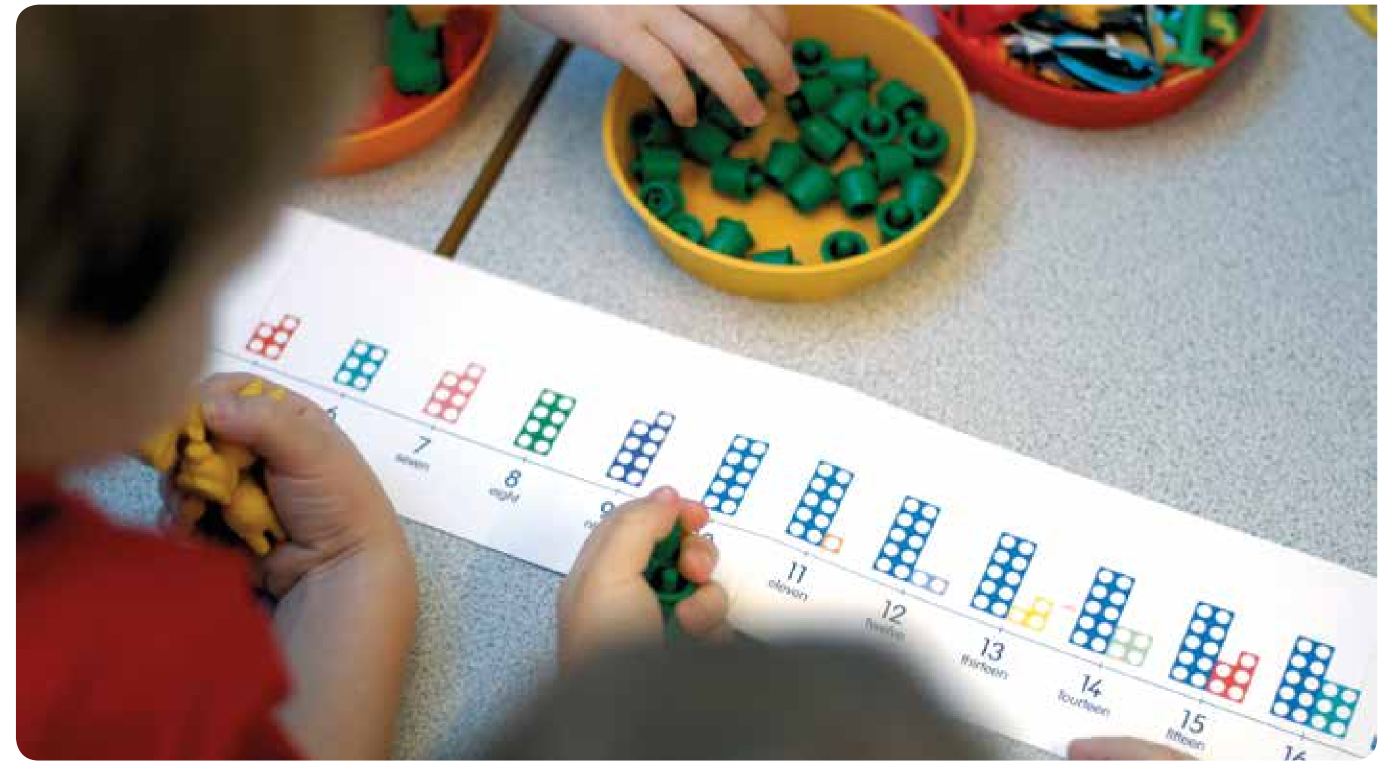
Assessing individual children's counting

Reviewing children's counting regularly helps keep track of how their understanding of the pattern of naming numbers in the number system is developing. This review has to be carried out individually but need not take long and can be part of the activities when teachers or teaching assistants are working with a focus group.

At each review, first of all establish the child's counting range by asking, 'How far can you count?' Then, choose a number from within the child's counting range and ask them to count on from that number (pronouncing 'teen' correctly).

As children's counting range extends, you will notice that counting across multiples of 10 often causes difficulty. To help, join in as they count across the multiples of 10 listening that they are pronouncing 'ty' correctly.

When children say that they can count to 100, check by asking them to count on from 19 to the mid-20s, from 29 to the mid-30s, and so on, until children have learnt the pattern for naming numbers to 100.



When children have securely established the forward counting sequence to 100, ask them to count back from 21 to the mid-teens, 31 to the mid-20s.

Continue to extend their counting range in whole class sessions modelling counting across multiples of 10 and multiples of 100.

When reviewing individual children's counting, check that they can count across the multiples of 100 by choosing a number (e.g. 198, 303) and asking them to count on or back from that number.

Even when children can recite numbers in order to 100, there are some further points of difficulty that have been widely observed in children. If these are left unresolved, they are highly likely to cause barriers to understanding in later years. These points should be checked again during the year in which children are working on Number, Pattern and Calculating 2:

- counting 0–90 correctly and then going back to 20 (confusing 19 and 90).
- counting to 100 correctly and then saying 200.
- counting to 109 and then 'drying up'.
- counting to 199 and 'drying up'.

This last point seems to be a particular sticking point for children aged 6 to 7 years. Many need help to cross 200 and encouragement to continue the count across the multiples of ten between 200 and 300.

When children feel confident with the counting sequence, you will find them saying, 'I can count to a thousand', or 'I can count to a million.' As before, choose numbers within their counting range for them to count on and back from, to check that they understand the patterns for naming numbers and can read and write these higher numerals.

If any child is struggling to remember the counting sequence to 100, check their ability to count up to 30 objects accurately one by one, progressing to 50 objects using a number line to place objects on or organizing objects into Numicon Shape patterns for support. You will find activities for this in the *Number, Pattern and Calculating 1 Teaching Resource Handbook*, Securing Foundations 1–3, Numbers and the Number System focus and practice activities.

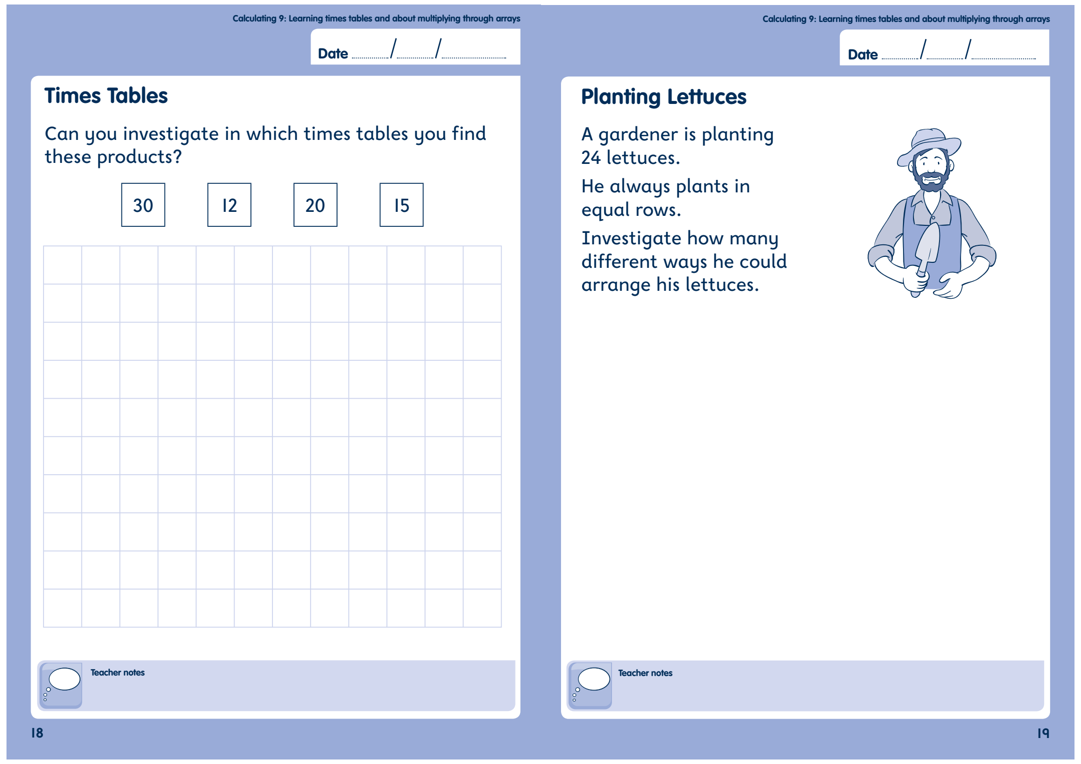
If you are concerned about other aspects of children's understanding, refer back to the progression of the Numbers and the Number System activities in the *Number, Pattern and Calculating 1 Teaching Resource Handbook*, Securing Foundations 4–8 and Numbers and the Number System 1–4.

What support is there for making summative assessments?

Assessment milestones and tracking children's progress

Within the medium-term plan for the *Number, Pattern and Calculating 2 Teaching Resource Handbook*, you will notice that there are milestones (summary statements) of specific points that children need to be secure in before they can progress to the next section of activity groups.

The statements in each milestone are founded on the assessment opportunities in the preceding activity groups and are also aligned to the national curriculum in England (2014). Your on-going assessing of each child will build up over the preceding period and you can keep a record of


Calculating 9: Learning times tables and about multiplying through arrays

Date ___/___/______

Times Tables

Can you investigate in which times tables you find these products?

30 12 20 15

Teacher notes

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Calculating 9: Learning times tables and about multiplying through arrays

Date ___/___/______

Planting Lettuces

A gardener is planting 24 lettuces.

He always plants in equal rows.

Investigate how many different ways he could arrange his lettuces.

Teacher notes

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each child's attainment and track their progress using the individual photocopy master of the collated milestones for the year (*Number, Pattern and Calculating 2 Teaching Resource Handbook*, photocopy masters 1a – 1d).

At the point of each milestone, you can reflect on each child's achievements and decide whether you need to plan further support and practice for them, giving them time to consolidate their understanding or whether they are ready to move on.

If children are moved on before they are ready, then their difficulties are likely to accumulate because they will not be adequately prepared for the new ideas they will meet.

Explorer Progress Books

Each activity group has two corresponding pages in one of the Explorer Progress Books. The tasks in the Explorer Progress Books have been designed to present children with tasks that give them opportunities to use the mathematics of the activity group. One page generally poses a problem that challenges children to use the mathematics they have been learning in the activity group within a new context. The other page generally aims to provide more open opportunities for children, enabling you to assess their ability to think mathematically more widely and also allowing you the opportunity to see the methods children use as they persist with an exploration.

The tasks are not tests, and so are as open as possible, inviting a full range of responses. Children should have available to them all the imagery and materials that have been available during the teaching of the activity group, and should be invited to express what they are doing as they do it. It is best to avoid affirming or denying anything a child says or does as they work. Instead, look and listen for what children do without your guidance.

In order to help you assess whether a child knows *when* to use the mathematics of the related activity group, the Explorer Progress Book tasks are designed to reveal as much as possible about the breadth of a child's understanding, so that you know what needs to be addressed in the future. These are not designed to be pass/fail tests, but are there to support you in assessing as accurately as possible children's current and/or retained understanding.

Give careful consideration to when children should be given each Explorer Progress Book task. You might ask them to complete one page at the end of work on the activity group, and then another two weeks later to see how much seems to be being retained. You might give children both tasks after the following activity group. You might give these tasks just before children face the next connected activity group. The point of these tasks is to discover children's understanding at a particular point – decide when the most useful point would be, in each case.

It may also be useful to keep notes about children's responses, and what you see as their significance for future work.

The Explorer Progress Books are designed to be given out to small focus groups, so that you can administer and monitor each individual child's responses to the pages. In this way, you will be able to build up a developing idea of each child's progress during the course of the school year.

What about formal testing for national authorities?

Formal tests and examinations are important hurdles for children and teachers, parents and carers, schools, universities, professionals, employers and governments. However, the nature of a formal test means that it tends to be an artificial and unique setting in which to 'do mathematics', and, in this sense, does not correspond with how children encounter mathematics in their mathematics lessons or everyday lives.

As a result, preparing for national tests needs time devoted to children's preparation for the uniqueness of the experience. Doing mathematics in the circumstances of a formal test or examination, however, should not become the paradigm for 'doing mathematics'. Children need to learn to function mathematically in a very wide range of situations.

Therfore, it is important not to confuse formal testing with 'doing mathematics' in any other situation. In a formal mathematics test, communicating is almost always severely restricted to written forms only; this allows for some imagery, but not usually for action with physical materials. Also, the written language common to mathematics test papers can be very formal. Children will need plenty of prior practice at interpreting such writing.

Children will also need plenty of practice at 'internalizing' their use of action and imagery, since physical models and screen-based imagery will not usually be available to them during a test. When using Numicon, there are many opportunities to encourage development of children's mental imagery, and children should be continually encouraged to 'imagine' actions, objects, movements and shapes as well as working physically, as often as possible.

Finally, think about how children will need to react when they encounter 'difficulty'. In their mathematics lessons, children will have been encouraged to express difficulty, to explain why something is challenging, and to use action and imagery to illustrate their thinking. Under exam conditions, children will have to communicate mathematically with themselves, work hard to express silently what the trouble is, use mental imagery, and thus respond positively to being 'stuck' in an exam.

Formal examinations and testing are a fact of life. They are important and children need to prepare for these unique events. However, it is also important to recognize that examinations are, by their nature, artificial and not representative of what 'doing mathematics' is about in any other situation.

In order to make ongoing assessments of children's understanding, allow them the full range of actions, imagery and conversation, and encourage them to communicate mathematically.

Key mathematical ideas

Underlying the activities in Number, Pattern and Calculating 2 are many key mathematical ideas that children will be developing and extending, as well as some conventions they may be meeting for the first time, such as 'empty box' notation.

In order to teach these ideas effectively, those who are working on these activities with children will need to be clear themselves about the mathematical ideas and about which activities address which ideas.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the activity groups of the *Number, Pattern and Calculating 2 Teaching Resource Handbook*. The concepts covered within this section will help with planning how to develop key mathematical ideas with children.

The introductory page for each activity group lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to the following section.

The maths coordinator may also find it useful to work on the key mathematical ideas in professional development sessions with the class teachers and the wider school staff.

In this section you will find overviews of:

Fluency, reasoning and problem-solving

The mathematics curriculum in England signals fluency, reasoning and solving problems as its aims, and the characteristics of Numicon match these directly. All three aims recognize mathematics as an activity (as we do), and that 'doing mathematics' essentially involves thinking mathematically in these specified ways.

Thinking mathematically is communicating mathematically with oneself, and Numicon is devoted to developing children's mathematical communicating – both with others and with themselves - through being active, illustrating, and talking.

Fluency is an attribute of communicating to do with smoothness of flow; it should not be confused with being fast at something. Understanding is involved in fluency, and a simple and mechanical calculating proficiency alone is not enough for children to be fluent.

Numicon develops richness in children's thinking/communicating through utilizing being active, illustrating and talking. This richness encourages children to respond flexibly to challenging situations, to have available a variety of means of communicating about (and hence thinking about) the structures and possibilities of new situations they are working in. Flexibility is what supports mathematical fluency in new and unfamiliar situations.

'Automaticity' necessarily underlies fluency as children progress to facing ever more complex situations. As every learner driver understands very well, many actions and responses need to become automatic in order to command a vehicle effectively in complex traffic situations. So it is with doing mathematics; the more automatic certain responses become, the more able to direct available attention to reflecting and acting in complex and new problem situations. With Numicon, practice developing familiarity and automaticity is integral to all activities.

Flexibility and automaticity together are what allow children quickly to assess any particular new calculation, as 'one that I could easily do this way'. In all the open discussion encouraged throughout Numicon, in work on 'non-computational thinking' and in the emphasis upon algebraic relations that underpin effective calculating, children are encouraged to approach calculating in a thoughtfully fluent manner, rather than mechanically. This is fluency based upon understanding, not upon mechanical repetition.

Although many people assume **reasoning** to be about using only words and symbols logically, in mathematics, reasoning is much wider than that. Imagery is involved in almost all mathematical thinking, and hence so are movement and action. The action, imagery, conversation, and relationships that are at the heart of Numicon constitute all the essential elements that will together develop into children's mathematical reasoning.

Even though all that might be on a child's page are written numerals and other symbols, the communicating that child has done or is doing with themself (i.e. their thinking) is almost bound to involve action, imagery and words as well. Even reasoning about abstract number ideas depends upon imagery such as number lines and patterns to communicate the 'logic' of the interrelationships involved.

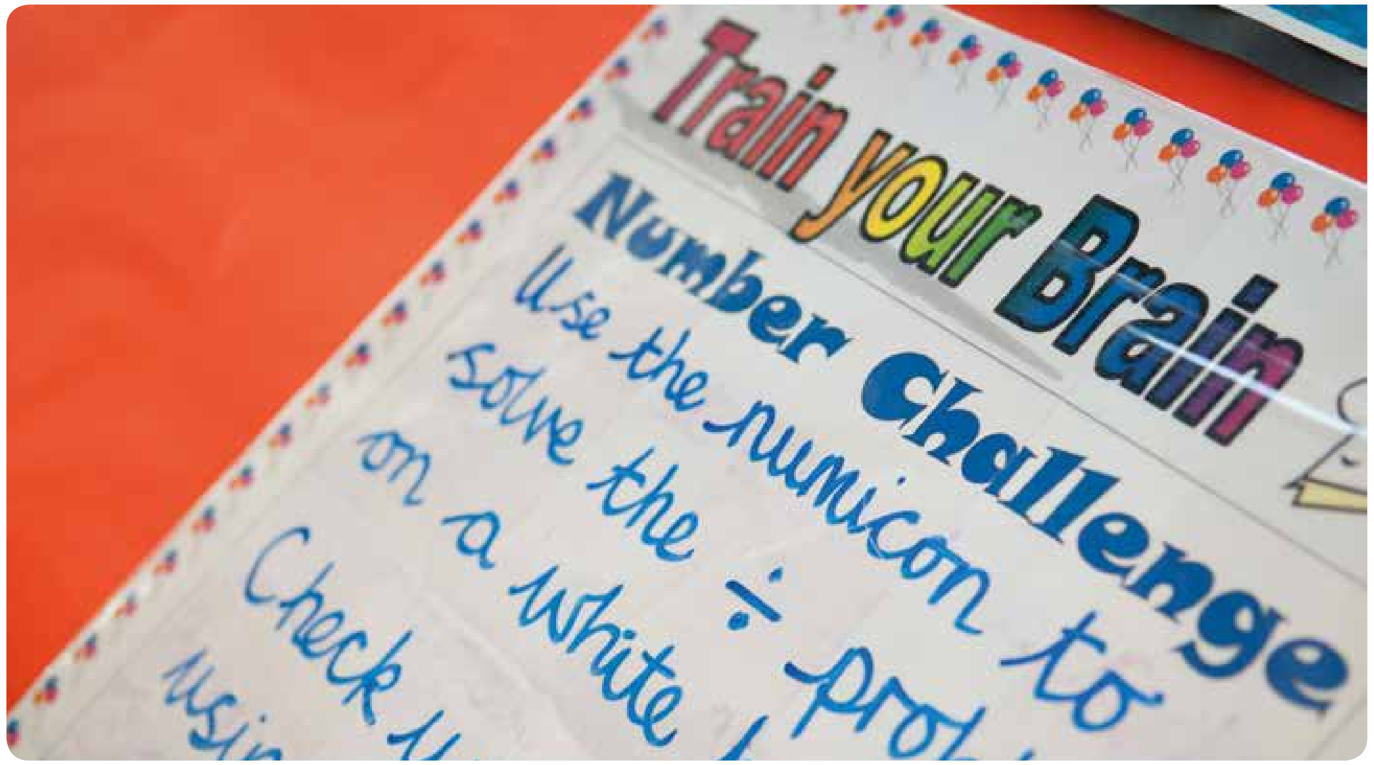
Following a line of enquiry, conjecturing relationships and generalizations, and developing an argument, justification or proof using mathematical language are integral to mathematics teaching with Numicon.

A fundamental point of learning to do mathematics is the ability to function mathematically in new situations – many of which will be unfamiliar or 'non-routine'. This again points to the need for flexibility, imagination, and also for courage – the kind of courage that grows in situations where children learn that challenge is normal, imagination is praised, and that perseverance is generally seen to pay off.

Using Numicon, there are three key things that help children develop the flexibility, imagination, courage, and persistence necessary to effectively engage in mathematical **problem-solving**.

Firstly, mathematics is grounded within contexts in which it is seen to be useful. If children can 'see the point' of the mathematics they are being asked to do, they are a good way towards knowing when that mathematics would be useful at other times.

Secondly, there is an acknowledgement that doing mathematics is challenging – unless children are in a routine situation that they have recognized. It is expected that children will find most Numicon activities suitably challenging, and that challenge will be normalized. When children are challenged – when they are 'stuck' – they know to try to communicate the difficulty they are experiencing as fully and richly as possible. This is, in effect, encouraging communicating when fluency has broken down. This is where the rich variety of communicating that is continually encouraged comes into its own; as children try to communicate their difficulty in as many ways as they can, new ways forward will almost always occur to them. The central message to children is that when they don't know what it is they need to do, they should be communicating mathematically to try and move forward.



Thirdly, in the Explorer Progress Books, children are regularly supplied with unfamiliar and quite 'open' situations that invite them to use the mathematics they have been learning. These experiences are designed both to confirm the message that challenge is normal, and to identify particular aspects of their work that will benefit from further attention.

By facing mathematical challenge, and by developing children's resilience and resourcefulness in the face of this challenge, children are effectively prepared to solve both routine and non-routine problems.

Fluency is developed through flexibility and automaticity, through developing an optimal variety in children's mathematical thinking/communicating to support their active engagement in new and unfamiliar situations. This same broad richness in thinking/communicating develops children's reasoning, their capacities to argue, conjecture, generalize and justify their conjectures. Finally, such communicative richness enables children's productive engagement in the widest range of new problem situations, and the emphasis upon normalizing challenge ensures that children are suitably prepared when they don't know what to do. Problem-solving becomes a familiar and satisfying experience.

Thinking mathematically

Numicon is aimed at developing children's mathematical thinking. To do this, the focus is on children's mathematical communicating, since thinking and communicating are essentially two sides of the same coin; we communicate with ourselves (i.e. we think) in the same ways that we communicate with others. Children's mathematical thinking develops as they learn to join in with the ways in which their teachers, as expert mathematical thinkers, communicate.

Effective teachers of mathematics have always recognized that learning mathematics is about more than learning facts and calculation techniques; it is about learning how mathematics is actually practised, i.e. about mathematics as an activity. Although children will learn about number relationships and calculations, they crucially need to learn how to go about **doing** mathematics in a variety of situations. There are several common and important elements involved in doing mathematics – in thinking mathematically in any situation – most of which stem directly from the essentially abstract character of many mathematical ideas. Most mathematical ideas are generalizations and, because of this, feel abstract.

The first key characteristic of mathematical thinking is an ever-present drive towards **generalizing**. It has been observed that a lesson that doesn't include children making at least one generalization is a lesson where they have not been doing any mathematics (Mason et al, 2005). Looking for patterns in every situation is central to helping children to generalize: once you have seen a pattern, you can generalize – know how it will continue – and thus predict. Making situations predictable is exactly what mathematics is about.

Encourage children to generalize by taking every suitable opportunity to ask, 'Do you think that will always happen (or always work)?', and by remembering that all generalizing involves stressing some features of a situation and ignoring others (*Ibid*), e.g. we help children understand what '3' means by showing them lots of collections of three objects, stressing how many there are, and inviting them to ignore the kinds of objects involved.

Another crucial element of mathematical thinking is to do with **reasoning** with generalizations. If there is no logical thought in the relating of generalizations to each other, the outcome of that thinking is likely to be faulty and is almost bound to result in misleading information. It is important to remember that children are faced with generalizations from very early on in their schooling, e.g. the counting numbers, and they are invited to 'be logical' about ideas like this almost from the beginning.

However, Piaget's work demonstrated that young children do not think logically in the way that most adults are able to; their gradual progress towards adult logical thinking is partly experiential, partly maturational. It is not possible to expect young children to think like adults. Having said that, Vygotsky showed how children's learning and thinking is never an isolated pursuit, but is essentially social; children are always learning to 'join in' with the ways in which the 'expert others' around them speak, think, and act.

Thus children's developing reasoning in mathematics should always be shared, explored, and reflected upon openly from their earliest experiences. There is no expectation of clear, formal, logical arguments from young children. However, there is a desire for them to notice that the way they think is important, and that thinking is worth thinking and talking about.

For children to feel willing and increasingly able to share and develop their mathematical thinking, they need to feel that their thoughts are welcomed, respected and important. If their thoughts are treated as simply 'wrong', low level, or if they feel ridiculed, then children will quickly close up and lose faith in their own thinking.

Central to Numicon is the self-confidence that children can learn to feel in their own mathematical thinking and reasoning; it is important to nurture this in a highly supportive atmosphere.

Every mathematics lesson also involves children working on, and with, relationships, i.e. working on structures. The best mathematics teaching occurs when the teacher and children's shared focus on the particular relationships (the structure) of a situation renders that situation more predictable for children. It therefore pays to be clear about the structures children need to be attending to in every lesson.

In Number, Pattern and Calculating 2, there is a continued emphasis on relationships and structures with an increasing range of structured materials, and also encouragement of what is sometimes called 'non-computational thinking'.

Non-computational thinking involves asking children to notice and to think about relationships between numbers in given situations *before* they rush in to calculate. This will be

of increasing help to them as they begin to realize they can actually change a given calculation – before they calculate – to make it easier. For example, 56 + 38 can be changed to 56 + 40 – 2. Such 'pre-calculating' thinking is also helpful to children as they face 'empty box' notation problems (see **Equivalence**) since the key idea involved is always equivalence itself.

Another essential characteristic of mathematical thinking lies in approaching situations **systematically**. Being systematic is both an aspect of effective reasoning and a secure foundation for generalizing (and thus predicting). Invite children to think systematically about possibilities whenever the opportunity presents itself. Get them to resist the impulse to guess wildly (as if they might somehow just hit on the answers magically) and encourage them patiently to explore the details of everything that could happen in a situation before reaching any firm conclusions.

Finally, an essential element in thinking mathematically is being able to **use and apply** mathematical generalizations and techniques in particular situations. There is, clearly, absolutely no point in children learning *how* to go through a lot of mathematical routines, or recite a lot of mathematical facts, unless they also learn *when* that mathematics and those facts can be useful. Numicon recommends that teachers always ground the mathematics they are teaching in contexts that make sense to children, and in which the mathematics involved is useful. Learning *how* is a waste of time unless children also learn *when.* In this regard, a sense-making context does not have to be 'real' or 'realistic', e.g. children will see some point in learning about partitioning numbers if that helps them do calculations more easily.

Mathematical thinking in these various ways is important throughout all the mathematical work children undertake. Whenever they are asked to do some mathematics,

whatever the topic, children should be encouraged to share their logic and reasoning in a supportive atmosphere; they should be encouraged to look beyond immediate examples and constantly ask, 'Will this always work?' In all their approaches to problem-solving, they should explore all possibilities systematically, and, every piece of mathematics they work with should be explicitly connected to sense-making situations.

Teaching in these ways will help children understand and learn what it is like to do mathematics – both learning *how* and learning *when*.

There are four key aspects to thinking mathematically:

- Generalizing. Being able to generalize helps children to predict. Much of mathematics is built on the ability to make predictions from generalizations.
- Reasoning with generalizations. If there is no logical thought in the relating of generalizations to each other, the outcomes of that thinking are likely to be flawed. Children need to be given every opportunity to develop their reasoning.
- Approaching situations systematically. Encouraging children to be systematic helps them reason effectively and also ensures they have a solid foundation upon which to generalize.
- Using and applying. Children need to learn when to do mathematics, not just how. Contextual examples help children to use the mathematics they are learning.

Pattern and order: essential for children of all ages

An essential idea underlying all Numicon activities is that of **pattern**. Pattern may not sound like a particularly mathematical idea because we are so used to patterns of one sort or another occurring in so many non-mathematical contexts. We could not learn to speak, for instance, without noticing patterns in the sounds we hear; patterns (rhythms) structure music and dance, and patterns in stories and plays enable us to anticipate the unfolding of a plot (and also, of course, allow our expectations to be manipulated by authors and writers). Much poetry crucially depends upon patterns to work its effect, and almost all scientific research is an attempt to discover or establish patterns in phenomena.

Importantly, it is the sensing of patterns in all our experiences that allows us to make essential aspects of our lives predictable, whether we want it so or not. Since seeing patterns is absolutely vital to much human survival, it is also something we do well. Seeing patterns is what enables us to predict what comes next and thus gain a measure of control over our environment.

A great many mathematical patterns involve an *order*, not just in a general sense in which we might talk about 'order vs chaos', but in the sense of *sequence*. For some mathematicians, such as Caleb Gattegno (who introduced Cuisenaire rods in the UK), the idea of sequential order is the first and most important of all mathematical ideas. Children meet order, the ideas of 'before', 'after' and 'next', very early in life. As they learn to count, children meet the order of numbers (one, two, three) long before they have any idea what those words mean. Everyday life is full of order situations – queuing in a shop, eating breakfast before lunch – and the simple idea of sequence lies behind many of the most important of all mathematical ideas.

Patterns are also essential to mathematics for a very special reason: they enable us to imagine actions going on 'forever' in our thinking, without physically having to do them. Counting is a good example. As we have a system for generating number names, we can imagine what it would be like to count forever without having ever actually to do it. Most people know they could count to one million, without ever having done it. This is thanks to the place value naming system; we know the patterns in number names which would enable us to go on forever should we ever be called upon to actually do it. Importantly, once children see the pattern that each *next* whole number is *one more* than the last, they know how the counting numbers go on forever.

The ability to think ahead, to see how things would be if they carried on like this is crucial to an understanding of our number system and to mastery of calculating.

Numbers form an infinite system and it would be impossible to do things like mental calculating without managing the infiniteness of numbers through patterns which go on forever. For example, noticing the pattern that, 'it doesn't matter which way round you multiply two numbers' (the commutative property) gives children the insight that they only have to remember half of their tables. Therefore, as they can predict this, they don't have to keep checking it out for every example.

It is impossible to overestimate the importance of pattern to mathematical thinking. In fact, a very large part of the most powerful branch of mathematics (algebra) consists of seeing, manipulating and generalizing from patterns. It is algebra that enables humans to launch space shuttles and bring them back successfully by predicting and generalizing from patterns. It is important to remember that in all key mathematical ideas discussed here, pattern and generalizing are essential.

In Number, Pattern and Calculating 2, children's experiences with patterns is extended to distinguish between repeating patterns and growing patterns. Repeating patterns may be illustrated either as linear or as cyclic. An example of a repeating pattern illustrated as a cycle would be the twelve hours of half a day as shown on a clock face, whereas the same repeating pattern illustrated linearly would involve a time line. Growing patterns occur where there is a regular increase of some kind; the staircase regularity evident when a set of Numicon Shapes or number rods are ordered from smallest to largest is an example of a simple growing pattern.

Names for children's first numbers: 'counting' and 'place value'

Counting is an essential activity for children to master, both because it is fundamental to an understanding of how we measure quantity, and because it is the key to understanding the generalizations we call *numbers*. An essential part of learning to count involves mastering an infinite set of names for numbers, in order. In this area, understanding what it is children take on when they try to understand the conventional ways we name our numbers is crucial.

Unfortunately, for children trying to learn number names, our society uses two parallel systems for naming numbers that don't always agree with each other. There is a written symbolic system (numerals) and a spoken and written verbal system (words) for saying and reading number names. These two systems would be hard enough in themselves for children to understand and remember, but as they often conflict with each other, learning them is a really tough task. For example, think about writing '18'. The '1' is written first, then the '8'. However, the 'eight' is said first when reading it as 'eighteen'.

Worse still, these systems conflict most often in the naming of the lower value numbers which children tend to meet first. This means children have to face the most confusing parts of the system at the beginning of their learning. The way names for numbers in numerals up to '50' are written and the way those number names up to 'fifty' are said are very different. There is the further problem, too, that children confuse similar sounding words, e.g. 'fifteen' and 'fifty'.

Consider the symbolic numeral code children are learning. Interestingly, civilization has been clever in devising a system for generating symbolic number names which not only allows for the inventing of new names forever, but which also shows instantly where in the series of number names any particular name is found. When reading '273' successfully, it is clear that it is the name of the whole number that comes next after 272. It is also clear that it comes a hundred before 373. This means that, unlike learning the alphabet, there is no need to remember every individual symbolic number name and its place in the order (which would be impossible anyway, since there are an infinite number of them); it is only necessary to master the system that generates the names.

The development of the symbolic number naming system we use today took the human race thousands of years, so it is not surprising if children can often take a while to master it. Once mastered, however, it can seem deceptively simple because understanding it has involved seeing a pattern, or rather several interlocking patterns. **Seeing a pattern is a very powerful thing to do, and seeing patterns is the key to doing most mathematics.**

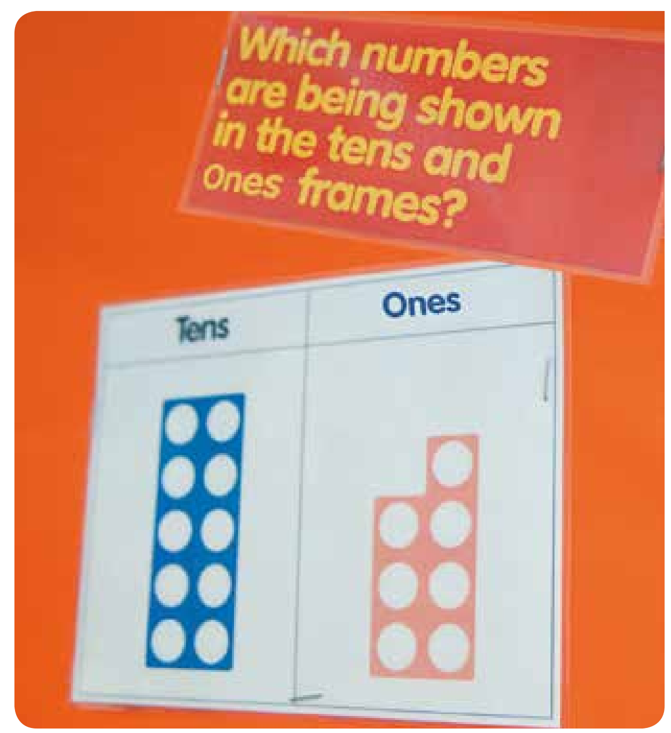
There are two essential keys to understanding the symbolic system for naming numbers. The first key idea is **grouping**. The number we call 'ten' (in numerals, '10') is the most

important number in our naming system because when counting collections, as soon as we have ten of something, we call them one of something else. So, ten 'ones' are called one 'ten', ten 'tens' are called one 'hundred', ten 'hundreds' are called one 'thousand'. In effect, in the language we use, we are always grouping things into tens (and then grouping groups of groups) to call them one of something else.

In children's early experiences of finding how many objects there are in a collection, it is always important to help them physically to group collections into tens as they try to find out how many things they have in front of them. Straightforward counting of objects one at a time can too easily become just a blind memory task for children as they struggle to remember which number name comes next. On the other hand, finding 'how many?' by grouping in tens (and then possibly tens of tens) reminds children constantly that our way of naming numbers uses a ten-based system, and this understanding is crucial for their later understanding of many calculating techniques.

The second key idea behind our symbolic system is called **place value**, signalling how the place of each digit within a string of digits signifies an important value. For example, it is the place of '2' in the number '427' that tells us it has a value of 2 tens, or 20. The term 'place value' actually refers to a symbolic code for naming and reading number names, and that children have to learn either to crack the code or to reinvent it for themselves (depending upon how they are taught).

Some people distinguish between what is called the 'column value' of a digit (the column value of '2' in '427' is '2 tens', because it is in the 'tens' column), and the 'quantity value' of a digit (the quantity value of '2' in '427' is '20', because



that is its value as a quantity). In Numicon, the important thing is that children understand that 'column value' and 'quantity value' are equivalent, e.g. that the '2' in '427' means both 'two tens' and '20', and that the two values are interchangeable. Children learn this equivalence through joining in conversations around place value – another instance illustrating the importance of conversations with children when teaching mathematics.

In the finding 'how many?' tasks that are used in Numicon, children are constantly encouraged to develop their counting skills. However, it is also important to recognize that their understanding of how ever larger numbers are dealt with, or named, is crucially dependent upon a grouping-in-tens system which uses a 'place value' code. It is vital that children understand that the number name 'thirty' does not just represent the number that is one bigger than 'twenty-nine', but that that number is being thought of as three groups of ten (represented very clearly in Numicon with three Numicon 10-patterns).

Therefore, when asking children 'how many?' questions, children should develop and remember the techniques not just of counting in ones but also of grouping in tens; this introduces the crucial place value code quite readily from their earliest experiences. Also, by finding the answers to 'how many?' questions on the Numicon 10s Number Line, children will make use of decade patterns which further reinforce the tens structure of our number naming system. Using the Numicon 10s Number Line also helps children **partition**, e.g. 36 as '30 and 6'(quantity value), which can sometimes, but not always, be a more helpful way of seeing the number when calculating mentally than the '3 tens and 6 ones' (column value) interpretation.

Understanding our symbolic number naming system, however, does not help younger children (or children with certain kinds of special needs) entirely understand our parallel verbal system of names for numbers. When saying, and reading, symbolic number names, it is often the case that what is being said conflicts with the normal rules for saying and reading. This confronts children with a number of confusing practices that, because they are only conventional (and not reasonable), children just have to accept.

The words 'ten', 'eleven' and 'twelve' do not signal that anything significant happens when counting beyond nine. This contrasts with the way that the symbolic code '10', '11' and '12' signals a shift from single numeral names (e.g. 9) to two numerals representing one number (e.g. 10). These three English number words conceal a crucial introduction to grouping and place value in the symbolic code that is not reflected verbally in English until reaching 'thirteen'.

There is also the fact that the 'teens' number names are said 'the wrong way round'. In the word 'sixteen', the 'six' comes first and is said first; in the numerals '16', 'six' is said first but is written second. The saying of teens symbolic numbers after twelve breaks the normal rule for reading in English from left

to right. To complicate things further, the morphemes 'thir-' and 'fif-' also obscure the 'three' and 'five' meanings normally attached to numerals.

In the words 'twenty', 'thirty', 'forty', and 'fifty' the morphemes 'twen-', 'thir-', 'for-', 'fif-', and '-ty' (being simply archaic speech corruptions) obscure the 'two-', 'three-', 'four-', 'five-' and '-tens' meanings which would reflect, respectively, the numerals of their symbolic representation.

Finally, there are conflicts in higher value numbers, e.g. between saying, 'two hundred and three' and the symbolic code '203'. Many children write '2003' for '203' because '200 and 3' is the way we actually say it. Children could be forgiven for feeling thoroughly exasperated by all these different instances in which the ways they are taught to say numbers contradicts the rules they are given for writing numbers in our place value code.

Compression: processes become objects

It is true that it is through counting activities that children begin to develop an early understanding of numbers. However, it is important to realize that their progress into calculating depends upon their going beyond counting to also seeing numbers as whole entities. This is sometimes described as moving from a process (counting) to talking about generalized objects (numbers). It doesn't help children to think of 'six' only as a routine of 'one-two-three-four-five-six' when the time comes to calculate; 'six' needs to also become a number object, an idea whole in itself, not thought of as a process of counting six individual little bits. For this reason, Numicon Shapes have been designed to help children see numbers as clear wholes; to see, e.g. 'six' as an organized pattern that is whole and complete in itself. It is as if children have to compress their understanding of the six 'ones' that make up the process of counting six, into a whole, composite number idea called 'six'.

The system of patterns used for Numicon Shapes is also designed to show children how each individual whole number relates to other whole numbers. The related structures of the Numicon Shape patterns (unlike domino or dice patterns) make a whole 'four' also look like 'one less than five', 'one more than three', 'two twos' and 'half of eight', whilst also allowing each distinctive pattern for two, three, four, five and eight to be entirely memorable in itself. It is the seeing of these patterns of relationships between whole numbers that are the foundations of children's later calculating. In the following example, the Numicon Shape patterns for one, two, three, four, five and eight are the images that will allow the following relationships to make sense to children without counting:

$4 = 5 - 1$ $4 = 3 + 1$
$4 = 2 + 2$ $4 + 4 = 8$

As suggested above, to help children go beyond counting procedures it is crucial that they learn from early on to find 'how many?' there are in collections without counting. This is again where pattern plays the key role. Numicon Shapes are designed to help children *see* how many things are before them by arranging them into systematic, recognizable patterns, thus again seeing numbers of things as organized wholes. Making recognizable patterns of objects enables children to see 'how many?' without counting; it enables them to compress lots of little unorganized pieces into an organized whole.

To the same purpose, coloured number rods are designed to encourage children to see numbers as wholes. That is why the rods that are used are not graduated into 'ones'. Work with these almost forces children to see numbers as wholes in relation to each other, since it is impossible to give any rod a value by itself – numerical values can only be ascribed to these rods by comparison with each other.

In the early stages of working with Numicon, these coloured rods are accorded number values in relation to the smallest (cream) rod – it being initially called 'one'. Later on, different rods will be called 'one', according different values to all the others. Compression, or seeing numbers as wholes, is achieved with number rods by encouraging children to give values to the rods on the basis of their relative positions within the whole series of rods.

There will be much more compression required of children later, as they gradually assimilate various aspects of fractions using Numicon. The way in which a common fraction is written is conventionally interpreted as both a process (that of dividing) and as an object (an amount); the fraction called 'two-thirds' is understood as both a dividing process (2 divided by 3) and as the outcome of that dividing – the number object called 'two-thirds' (see **Fractions**).

In mathematics, it is quite common for processes to become objects, and this compression, signified with the symbols used, normally takes quite a while for children to appreciate. Much depends upon how children are engaged in dialogue about these moves, and the imagery and models used to support these conversations. However, the most important thing is to allow children plenty of time to accommodate to such key shifts in communicating.

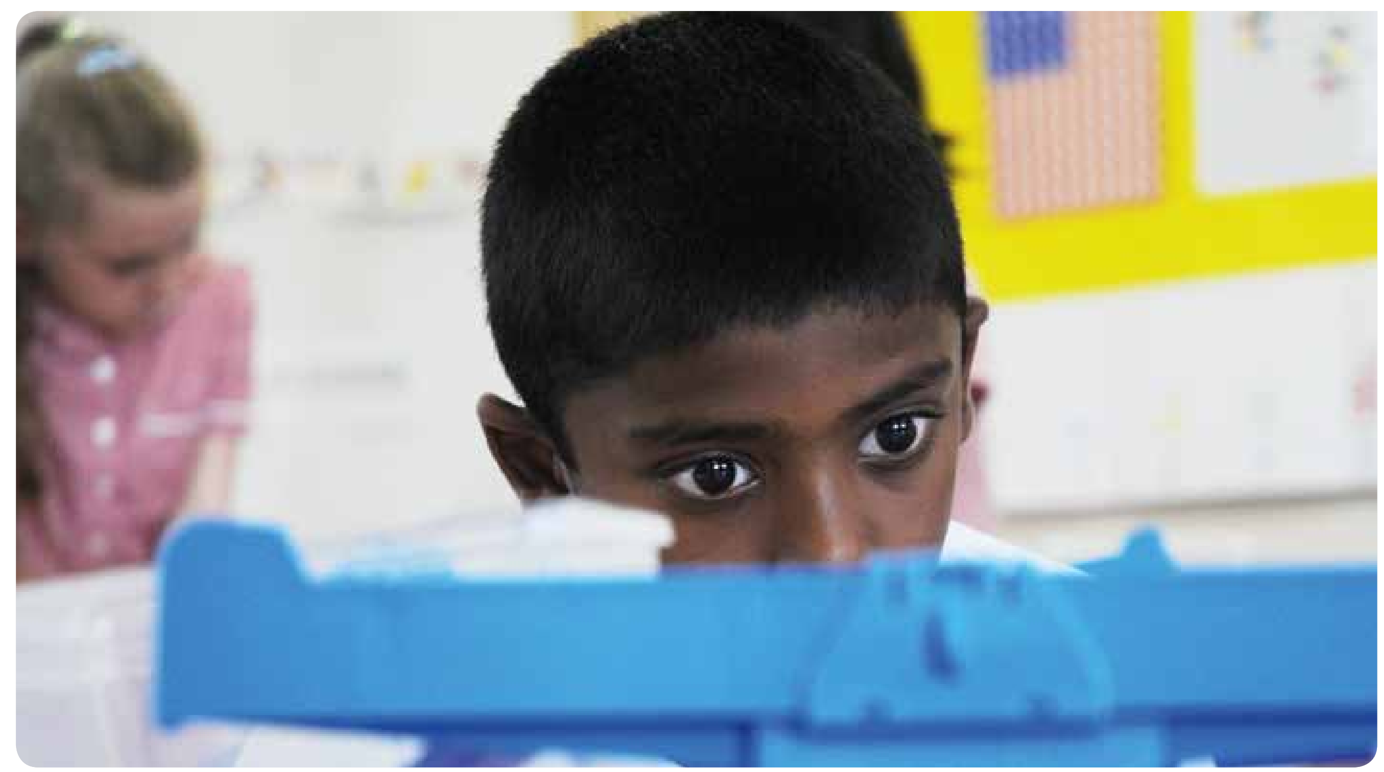
Equivalence

Equivalence is one of the most important mathematical ideas. It is important, when working with young children, to draw explicit attention to it. As indicated when talking about compression and counting, children actually work with equivalence from very early on in their thinking. Equivalence is also the big idea behind the conservation judgements children make. In fact, it is difficult to see how children could have made any progress in their learning of language (long before school) without noting various equivalences in the sounds and expressions they have heard and made.

Equivalence literally means 'equal value'. The most interesting and important instances of equivalence occur when two or more things are of equal value, but look different. In early calculating, there are three occasions when children face instances of equivalence explicitly: quantity value and column value (see **Names for children's first numbers – place value**), the introduction of the = sign (which means 'is equivalent to') and later, when they meet fractions ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ …). However, there are numerous other occasions when they explicitly need to see equivalence.

For their mental arithmetic strategies to make sense, children have to be able to see what are called the decompositions of any number as equivalent to each other, e.g. $9 = 1 + 8 = 3 + 6 = 10 - 1$. In measuring, equivalences between units (100 cm = 1 m, 1″ ≈ 2.54 cm) are at the heart of being able to understand systems of measures and relations between them.

When young children seem not to understand something in arithmetic that might be deemed to be clear and straightforward, there is often an equivalence we are seeing



that they are not, or vice versa. Equivalence is about things that are worth the same but which look different. Very often we find it difficult to see beyond appearances. It is common for children to hear that digits in the tens column refer to 'tens' and not 'ones'. Yet, when explaining decomposition, children also hear that a 'ten' is as good as ten 'ones'. The digits involved all look the same, but sometimes 'tens' are not 'ones' and sometimes they are. What enables this interpretive switching is an underlying understanding of the equivalence between one 'ten' and ten 'ones'.

Importantly, there are different 'structures' to situations requiring arithmetic that have an equivalence to each other. Although two situations may look different, they can be 'equivalent' in that they invite the same operation. For example, 'finding a difference', looks different to a 'take away' situation, yet they both invite subtracting.

There is also a language problem here. It can be difficult to use the word 'equivalence' with children without resorting to an easier word – 'same'. When talking about equivalence, it's common to hear mention of it being 'the same thing'. Unfortunately, equivalence does not mean 'the same' – it means **equal value, different appearance**.

In Number, Pattern and Calculating 2, children's understanding of equivalence is developed significantly by practical use of a simple pan balance, with which the proportional weightings of both Numicon Shapes and number rods illustrate physically how, e.g. 3 + 2 'balances' (=) 1 + 4. This multi-sensory illustration of equivalence is exploited carefully as children are introduced to 'empty box' problems, e.g. ☐ + 7 = 13.

Arithmetic operations, or 'the four rules': adding and subtracting

Number, Pattern and Calculating 1 was principally concerned with introducing two arithmetic operations: adding and subtracting. The foundations of multiplying and dividing were also being laid in counting on (and back) activities, but the operations themselves were not introduced explicitly.

In Number, Pattern and Calculating 2, in relation to adding and subtracting, children learn more about: structures (the forms in which adding and subtracting situations occur); methods (how to do the calculations); and properties (characteristics and relationships of adding and subtracting to each other).

Structures for adding and subtracting

There are usually thought to be two adding structures: **aggregation** and **augmentation**.

Aggregation is putting together. Two or more amounts or numbers are put together to make a total or sum, e.g. 'I had five pounds. John gave me ten pounds. How much did I have in total?'

Augmentation is about increasing, typically, when one amount is increased or made bigger, e.g. 'Special offer! One third extra, free!'

With subtracting, there are usually thought to be four structures: **take away**, **decrease**, **comparison**, and **inverse of adding**. Already, it is clear to see one big reason children find subtracting more difficult: it is much more complicated.

Take away refers to those situations where something is lost, or one thing is taken away from another, e.g. 'Gemma has six sweets. She eats three. How many does she have now?'

Decrease is about reduction, e.g. 'Special offer! A third off!'

Comparison is where two amounts are being compared and we want to find the difference, e.g. 'Samir has 10p and Nihal has 12p. What's the difference between the amount of money Samir and Nihal have?'

The **inverse of adding** structure concerns wanting to know how much more of something we want or need in order to reach a particular target, e.g. 'Those blue shoes cost £10. I have £5. How much more do I have to save up to buy those shoes?' Children can often feel very confused about adding on in order to accomplish subtracting, and it is important to be clear about what is going on here. The reason this adding manoeuvre is included as a subtracting structure is because the adding on in these cases is done in order to find out a difference. Contrast this with normal adding where children know how much to add, and they do it.

Mental methods for adding and subtracting

The mental methods for adding it is important for children to learn involve two elements: fluent recall of basic number relations (number facts), and strategies that use these relations to reach new results. The simplest way of thinking about these two elements is that recall involves a one-step response to an adding problem, whereas a strategy involves several steps. Some methods are almost one-step for some children; some reasoning will be almost instant and will therefore feel like almost one-step.

The one-step responses children should be able to make in Number, Pattern and Calculating 2 involve remembering the basic adding facts for all numbers up to ten (especially doubles), adding one, two, ten and zero to any number, and the corresponding facts to twenty (e.g. 3 + 7 = 10 so 13 + 7 = 20).

Importantly, because children need to move beyond counting for arithmetic, Numicon does not use counting on as a method of adding; in Numicon, adding is what is done

instead of counting. The strategies for children to master range from exploiting near doubles, through bridging, to adding more than two numbers. It is particularly important for children to generalize as a strategy. A key generalization for children to make in Number, Pattern and Calculating 2 is to be able to generalize from single-digit number facts (e.g. 2 + 4 = 6) to adding whole tens (e.g. 20 + 40 = 60).

The position is similar with subtracting. The corresponding subtracting facts need to be remembered for one-step responses (e.g. if 4 + 5 = 9, then 9 – 5 = 4), together with strategies for reaching new results like taking away and finding the difference. In Number, Pattern and Calculating 2, children should be encouraged to generalize from number facts to ten to number facts to whole tens (e.g. from 8 – 5 = 3 to 80 – 50 = 30). Once again, Numicon does not use counting on or back as methods of subtracting; subtracting is what is done instead of counting.

In Number, Pattern and Calculating 2, emphasis is placed upon adding and subtracting up to 100. Initially, materials and imagery are used so that children have the opportunity to internalize the actions and imagery involved, and work mentally.

Towards written methods of adding and subtracting

When tackling more and more complex calculations, it can become too difficult to hold all the individual numbers involved in a calculation mentally. It is at this stage that 'column' written methods first become useful. Children are prepared for these in Number, Pattern and Calculating 2 through introducing vertical recording of some calculations. Such vertical (column) recording is particularly helpful when children are partitioning into tens and ones to calculate.

Trio number and 'part-whole' relations – numbers come in threes

There is good evidence (Fuson, 1992) to suggest that as children learn to add and subtract, their thinking begins to form part-whole associations between related sets of three numbers, e.g. between 3, 5 and 8. They somehow begin to remember and connect these number threes together in part-whole relationships that both relate inverse adding and subtracting facts and (it is suggested) help them to derive near-double relationships. For example, built trio associations ('triad numbers', as Fuson refers to them) between 4, 4 and 8 (4 and 4 are partner parts of the whole 8) allow children to connect the adding relation to the inverse subtracting relation 8 – 4 = 4, and to calculate that, 4 + 5 must therefore equal 9 and 9 – 4 must be 5.

Since understanding part-whole relationships and remembering the sets of number trios that we usually call basic adding and subtracting facts are so fundamental to children's progress in effective calculating, in Number, Pattern and Calculating 1 emphasis began to be placed upon these number trio associations in explicit ways with

both Numicon Shapes and number rods. In Number, Pattern and Calculating 2, these basic associations are called trios, and children are introduced to specific forms of illustration to support their development of fluent recall in the basic adding and subtracting number facts.

It is a good idea to take every opportunity to draw attention to part-whole relationships throughout the curriculum. Missing pieces of puzzles, a child away from school, sharing a whole cake are all situations that can be used generally to develop an awareness of part and whole relationships that are also essential to understanding many number relationships – not least, of course, fractions.

Arithmetic operations, or 'the four rules': multiplying and dividing

Two principal and relatively new ideas are formally introduced to children in Number, Pattern and Calculating 2: multiplying and dividing. Foundations for both of these ideas will have occurred much earlier in children's experiences and in the 'counting on (and back)' , doubling and halving and fraction work in Number, Pattern and Calculating 1.
It is important to realize that this explicit work on multiplying and dividing is an important new stage in children's introduction to what is often called 'multiplicative thinking' or 'proportionality'. This is a very significant cluster of ideas (including fractions) that are essential to children's future progress.

To think about proportionality further, consider basic ways in which static quantities may be compared with each other.

Imagine two different lengths, say 2 m and 6 m. They can be compared in two basic ways: it could be said their difference is 4 m, or it could be said that 6 m is three times as long as 2 m (the 6 m and the 2 m are in the ratio 3:1). The first comparison is 'additive', and the second is 'multiplicative'. It could also be said that 2 m is 'one third' of 6 m (or the 2 m and the 6 m are in the ratio 1:3), which is also a 'multiplicative' comparison. Scaling down (one third of) makes it clear how fractions are involved in multiplicative comparing; it is possible to multiply by fractions less than one as well as by numbers bigger than one. This may also help illustrate the understanding that fractions *are* ratios.

More importantly, the world is constantly changing, and understanding how one thing is changing in relation to another is often crucial to experiences. The speed at which a car is travelling is the ratio between the time it is taking and the distance it is covering; the steepness of the hill is the ratio between the vertical height being gained and the horizontal distance being covered. The rate at which prices are increasing in relation to wages, and interest rates, all make a lot of difference to economic well-being. It is the fact that ratio comparisons underlie so much everyday experience in the physical and social worlds that makes 'multiplicative thinking' or 'proportionality' so important for children to master.

Multiplicative thinking involves multiplying, dividing, fractions, decimals, percentages, ratio and proportion, all related to each other. It is vital that children understand how these ideas are connected and, therefore, that such ideas are never taught as if they were completely separate topics. This is obviously a very complex area, and children need to be

allowed plenty of time to ask their questions and explore situations in which ratios are central to understanding. At this stage, these new steps are crucial.

Multiplying

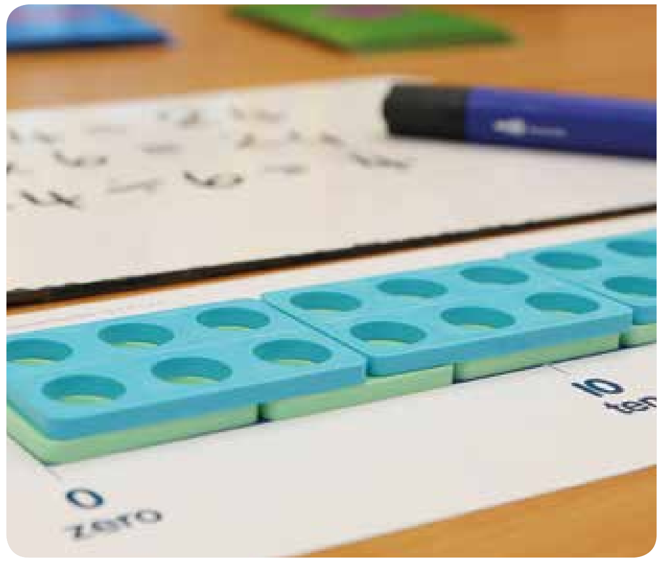
The idea of multiplying has many aspects, not all of which will make sense immediately to young children. In Numicon, distinction is made between the repeated adding, the ratio (or scaling) and the array (or correspondence) structures of multiplying.

Repeated adding occurs in situations where equal amounts are added together, e.g. 'Five tables each need six place settings. How many place settings are needed altogether?'

The ratio structure occurs where amounts are 'scaled up' to meet a new demand, e.g. when twice as many people turn up for dinner, and a recipe has to be 'doubled'.

The array structure is used in Numicon primarily as an illustration that helps children understand the many properties of multiplying, but it is useful in its own right in situations where 'all possibilities' might need to be found, e.g. '12 girls and 12 boys are learning to dance; how many possible couples are there?' (This is technically called the 'Cartesian product'.)

In Numicon, multiplying is introduced to children through the repeated adding structure, since this is intuitively understandable and builds usefully upon the children's experiences of counting on in twos, threes, fives and tens from Number, Pattern and Calculating 1. Repeated adding is the familiar 'so many lots of something' idea, and it has an important inverse connection to the 'how many x's in?' structure of dividing (formally called **quotition**). As a result, multiplying is first introduced to children in Number, Pattern and Calculating 2 through the repeated adding structure. Later on in these activities, once children have an understanding of the repeated adding structure, they are introduced to the array structures. Technically, children



will also have been introduced to the scaling structure in Number, Pattern and Calculating 1 as they halved and quartered amounts, but this is 'multiplying by a fraction less than one' and so was not introduced to children explicitly as 'multiplying'.

Importantly, in multiplying, children will be meeting a situation in which two numbers are multiplied together to give a product and, in many situations, each number plays a different role – one number refers to an amount being multiplied (technically, the multiplicand) and the other determines how many times that number is to be multiplied (the multiplier). One number is being multiplied; the other does the multiplying. This is subtly quite different to the aggregation structure of adding, in which all numbers play the same role as each other; at this stage, when multiplying with the children, there is no explicit statement about the difference in the roles of numbers. In some array situations, such as finding all possibilities, the two numbers have identical roles and they are simply 'multiplied together'.

When reading and recording multiplying sentences (e.g. $4 \times 7 = 28$) there are many possibilities of interpretation, and often a surprising amount of controversy about whether '4×7' really means 'four sevens' or 'seven fours'. Of course, the array structure quickly demonstrates that their product is the same, but some feel that only one reading of the sentence can be 'mathematically correct'.

However, there are choices and there are good reasons for choosing either way. In Numicon, an expression such as 4×7 is introduced as being read as 'four times seven' (four 'lots of' seven) in order to exploit everyday use of the word 'times' (signalling repeated actions), to tie-in with the traditional reading and saying of multiplication tables in the UK; to be consistent with measuring unit conventions, e.g. 3 kg or three 'lots of' a kilogram; and to be consistent with later algebraic expressions, e.g. $4x + 3y$).

Finally, in order to establish equally essential links, the images and patterns developed with multiplying in Numicon are very quickly exploited to introduce dividing.

Dividing

There are three essential structures of dividing: the grouping, the sharing, and the ratio (or scaling) structures.

The grouping structure (quotition) occurs in situations where an amount is known , the dividend, and we want to know how many times a different amount, the divisor, will 'go into' the dividend. This type of situation will lead to remainders when the divisor is not a factor of the dividend, e.g. 8 'goes into' 43 five times, leaving a remainder of 3. This is often read as '8 divided into 43' (or '8s into 43').

The sharing structure ('partition') occurs in situations where again a known amount is to be shared (the dividend). This time it is known how many equal parts the dividend is to be shared into, but how big each 'share' will be is currently unknown. This type of situation can lead to fractions when the number of shares (the divisor) is not a factor of the dividend and the object(s) being shared can be broken into parts, e.g. three chocolate bars shared between two people will give each person one and a half bars. We might call this '3 divided into 2 parts' (hence *part*ition) or '3 divided by 2'.

It will later be important that children learn to distinguish between these two types of situations if their answers to dividing problems are to make sense. If there is a situation where there is a known amount of money, e.g. £32 (the dividend) and the question is to find out how many tickets at £1.50 can be bought with the money, the answer is 21 with a remainder of 50p. Not 21.33333… (as a calculator would show). This is a grouping problem (quotition); the aim is to know 'how many times £1.50 goes into £32' – there is no sharing involved and fractions make no sense as an answer. This is £1.50 into £32.

On the other hand if 3 children are going to share 10 fish fingers (the dividend) between them (fairly!) their equal shares will be 3.3333… (three and one third) fish fingers each. In this situation (partition) it is already known how many parts the fish fingers are to be divided into (3), but it is not initially known how big the resulting equal parts will be. This is 10 shared (or divided) into 3 equal parts.

Unfortunately for children (once again), the language used when speaking about dividing calculations is often confusing. The word 'into' tends to be used for both sorts of situation – dividing 3s 'into' 10 (quotition), as well as sharing 10 'into' 3 (partition). It is also common to hear of dividing 10 'by' 3. And to complicate things further, when introducing mathematical symbols for dividing to children, it is easy to imply different structures for the calculation. For instance, 10 ÷ 3 tends to be read as 'ten divided by three', and is often explained as a

'sharing' problem, whereas $3\overline{)10}$ tends to be read as 'threes into ten', probably because the numbers in this second case appear in the reverse order and the desire is for children to use times tables as they solve it.

In both cases, the aim is for children to 'do the same dividing', but these symbols are often explained as two quite different structures of dividing (the first partition, and the second quotition) and children can very reasonably struggle to understand exactly what it is they are being asked to do. Should they share 10 into 3 parts (partition), or should they find how many 3s in 10 (quotition)?

Eventually, children will come to understand that dividing can be seen either way and that whichever way a dividing problem is seen will affect the kind of answer given. Mixing up both structures of dividing without making them distinct to children often leads to confusion, and to answers that don't make sense. In particular, children often struggle to understand what to do with remainders.

Sometimes a dividing calculation involving integers doesn't work out to give a whole number solution. Dividing the dividend by the divisor shows 'how many times' the latter 'goes into' the former, but there is a small whole number 'left over'. Children are often confused about what to do with the left over number; sometimes they leave it as a remainder, and sometimes they carry on dividing the remainder into fractions (or decimals). When should they do which? The difficulty for children is that there are two quite different reasons for leaving a remainder, and a third reason for going into fractions/decimal fractions.

The first reason for leaving a remainder is when dealing with a quotition situation, and fractions would not make sense. For example, if 23 people need taxis home and a taxi will take five people, divide 5 into 23. The solution to this problem is not that four and three-fifths taxis are needed, but that hiring four taxis will leave a remainder of three people unable to get home. Therefore, it would be better to order five taxis.

The second reason for leaving a remainder is totally different. In a partition situation, there may be a need to share out objects which cannot be broken into smaller parts. For example, if 3 children have 50p to share between them, they can have 16p each and there will be 2p left over that cannot be broken down into three equal parts – it is, therefore, a remainder.

The reason for continuing to divide a left over whole number into fractions occurs in a partition situation where what is being shared can actually be broken down into smaller parts. For example, if 6 people are sharing 4 pizzas (4 ÷ 6 as a dividing problem), the solution is not 0 remainder 4 pizzas (everyone gets nothing), but that each gets two-thirds ($\frac{4}{6} = \frac{2}{3}$) of a pizza.

Children need a lot of experience with all three kinds of situations. In Number, Pattern and Calculating 2, dividing is first introduced as grouping (quotition), while sharing (partition) and fractions resulting from a dividing calculation

are left for later years. Simple fractions are, of course, discussed and used with children (see **Fractions**), but not yet as ways of dividing up whole number remainders in dividing calculations.

Dividing is firstly and distinctively introduced as the grouping structure, i.e. as 'undoing' multiplying. Having shown how 7 × 3 = 21 in multiplying, this is then connected with dividing as grouping (quotition) by asking, 'If 7 times 3 is 21, how many 3s are there in 21?'

Finally, the ratio structure of dividing appears in Number, Pattern and Calculating 2 with more halving and quartering. In effect, this is multiplying by a half, a quarter, but children will not be asked to make this explicit conceptual connection until much later.

Methods of multiplying and dividing

Children need to learn multiplication tables and in Numicon this is gradually developed in a carefully organized sequence, exploiting all available patterns to ensure these essential number facts are both connected with each other and are memorable. An essential element in children's calculating fluency comes from rapid recall of their times tables facts.

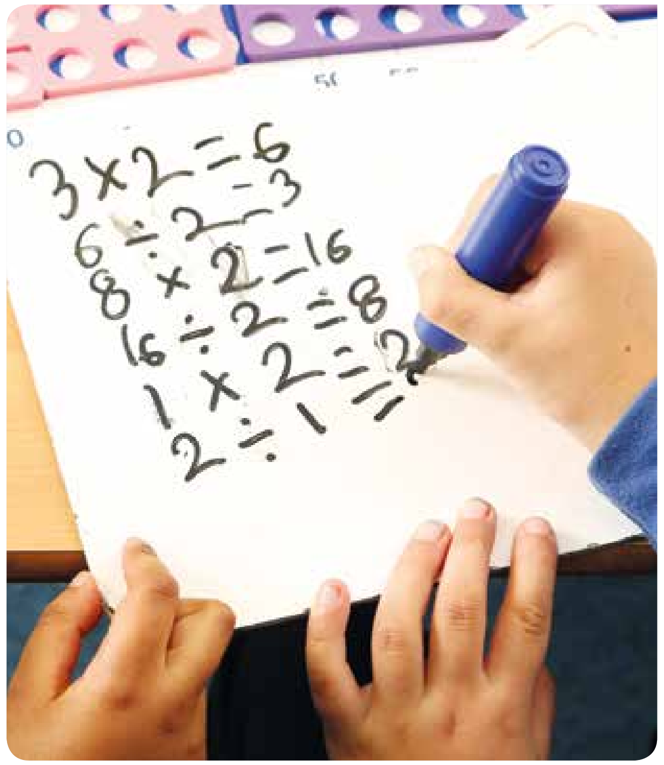
In Number, Pattern and Calculating 2, children rehearse counting on in twos, threes, fives, and tens, and are introduced to the corresponding 2, 3, 5 and 10 times tables. Written 'column' methods of multiplying and dividing are developed in Number, Pattern and Calculating 4.

Properties of the four operations: making mathematical connections

Adding, subtracting, multiplying and dividing are collectively called 'operating' on numbers. Performing each 'operation' on two numbers produces a third number as an outcome.

Just as adding and subtracting were introduced closely together, multiplying and dividing are introduced closely together. The purpose is to offer children the richest opportunities to make important conceptual connections between all four arithmetic operations.

When thinking about connecting these ideas is to important to consider how these four operations work in practice. Getting used to doing calculations using the four operations, tends to cause children to notice particular features. It's important to then generalize what they have noticed, usually without bothering to give such features a name. For instance, children might notice that it doesn't matter which way round they do any adding or any multiplying, and then use that observation to calculate 58 + 3 rather than 3 + 58 or 2 × 5 rather than 5 × 2. While they may do that in practice, they don't explicitly call it the commutative property of adding.



However, the ideas about how these arithmetic operations work are actually quite important algebraic ideas (they are about structure) and consequently within mathematics they have formal names. Whether or not it is important for children to know and use these mathematical names, it is important for their mastery of calculating that children understand the properties of adding, subtracting, multiplying and dividing to which they refer. It is important to monitor children working with the ideas.

Inverse

Adding and subtracting have what is called an inverse relation to each other. What this means is that they can each *undo* the other. If 6 is added to a number, that adding can be undone by subtracting 6, and vice versa. This knowledge is important to children for several reasons. First, the more connections they can make between things they learn, the more meaningful their learning is. Second, it is important children don't think adding and subtracting are completely unconnected because if they do, they will never understand the inverse of adding structure of subtracting. Finally, children should understand that an adding or subtracting calculation can always be checked by doing an inverse calculation.

Multiplying and dividing also an inverse relation to each other. Noticing how dividing undoes multiplying (and vice versa) is crucial to connecting these two operations to each other. This will also help children see that if multiplying is seen as repeated adding, it can make sense that dividing be seen as repeated subtracting (quotition). These are important foundations for the extended multiplying and dividing calculations children will learn later.

As mentioned above, in relation to counting and place value, partitioning numbers in various ways is the inverse of the making of numbers that children did in forming the early number facts and in answering 'how many?' questions, without counting, by grouping objects in 10s.

Finally in relation to inverses, empty box notation is used from Number, Pattern and Calculating 2 onwards to prepare children for the idea (in algebra) that letters are often used to stand for amounts that are unknown. Children are first learning to use symbols to stand for unknown amounts by using empty boxes, e.g. ☐ + 7 = 13 where they are asked to work out which number should go in the empty box to make the number sentence true. This type of problem again invites children either to undo a number fact they can remember, to subtract 7 from 13, or (physically) to see what needs to be put into the left-hand side of some scales containing 7 in order to 'balance' 13. In either of the first two cases, they are again using an inverse action; in the latter approach they are exploring equivalence.

Commutative property

Adding has a commutative property; subtracting does not. It doesn't matter which way round an adding calculation is done, but it does matter when doing a subtracting calculation, e.g. 12 + 6 = 6 + 12, but 12 – 6 does not equal 6 – 12. Similarly, multiplying has a commutative property; dividing does not. This makes sense since multiplying can be seen as repeated adding, and dividing as repeated subtracting.

Associative property

If there are three numbers to add together, it doesn't matter which pair are added first before then adding the third, e.g. with 2 + 3 + 5, the 2 and the 3 can be added first (and then the 5), or the 2 and the 5 first (followed by the 3), or the 3 and the 5 first (then the 2). Whatever way round the calculation is done, the answer is always the same: 10. Because of this, adding is said to have an associative property. The same applies to multiplying.

Is the same thing true when dealing with subtracting calculations? Take the example of 12 – 4 – 1. If subtracting had an associative property, it wouldn't matter the order in which the subtracting was done, the answer would always be the same. Mathematical knowledge says that this isn't the case. It is possible to subtract 1 from 4 and then subtract that number from 12. This would leave us with 9. Alternatively, subtract 4 from 12 and then subtract 1. This would leave an answer of 7.

Later, children will learn about the use of brackets to solve this communication problem, but for now they need to be aware that it does matter which two of three numbers are subtracted first. Subtracting does not have an associative property. Dividing does not have an associative property either.

Zero, and doing nothing

Most children notice that there's something unusual about zero. Within adding and subtracting, zero is what is called an identity element. This means simply that operating with it leaves everything exactly as it was; adding or taking away zero amounts, in effect, to doing nothing. Children need plenty of help understanding this because (quite rightly) they can't see the point of doing nothing. There is no point – it's just that zero is a number (it has its own position on the number line) and it can be added and subtracted. It just gives a strange result: no change at all. In multiplying, zero behaves like a rogue element, destroying everything it touches. This reinforces for children that zero really is something very strange.

Progression in mental calculating: adding and subtracting whole numbers

Users of Numicon should be aware of the rationale for developing the difficulty of mental calculating with children. Almost all approaches to teaching calculating begin with children learning single-digit number combinations. Thereafter, however, for calculations involving 2-digit numbers, widely different developments occur in different

teaching approaches, mainly due to the inclusion (or not) of *counting on or back in ones* strategies.

In Numicon, counting and calculating are deliberately developed separately. The relationships between whole numbers (that children see illustrated and feel physically with Numicon Shapes and number rods) are used for calculating, thus obviating any need to teach laborious and inefficient counting in ones strategies which children later cling on to.

Until relatively recently in the UK, before the introduction of a National Curriculum in 1988, young children traditionally learned written calculating from their earliest years in schools. The development of calculating difficulty for children was, in those days, a simple matter – one first learned tens and ones without 'carrying' or 'borrowing', and later with 'carrying' and 'borrowing'.

As such, there have thus been no traditions in the UK regarding the early teaching of mental arithmetic. When trialling Numicon, the practices and experiences of other European countries were closely monitored. In those countries, mental arithmetic has commonly been taught before written calculating. Numicon has adopted the following development of difficulty, comprising eight carefully sequenced stages of progression. This sequenced approach has proven to be effective in practice, supported with the parallel sequenced approach to teaching numbers and the number system.

Type		*Example*
O ± O		3 + 5, 8 – 4
T ± T		60 + 20, 60 – 10
T ± O		40 + 6, 30 – 5
TO ± O	without crossing tens boundary	53 + 4, 46 – 4
TO ± O	with crossing tens boundary	53 + 8, 33 – 5
TO ± T		38 + 40, 64 – 20
TO ± TO	without crossing tens boundary	73 + 52, 59 – 24
TO ± TO	with crossing tens boundary	46 + 37, 53 – 28

This sequence of eight stages is more finely graded than the earlier traditional approach to written calculating and is particularly well suited to the challenges of mental calculating.

Fractions

Fractions involve complex ideas, and children will eventually meet several different complex symbolic ways of representing what are essentially the same numbers (decimal fractions, ratios, percentages, proportions). Number, Pattern and Calculating 1 and 2 together provide an introduction to fractions. In Number, Pattern and Calculating 2, conventional notation for common fractions is introduced formally through visual association with the conventional signs for division.

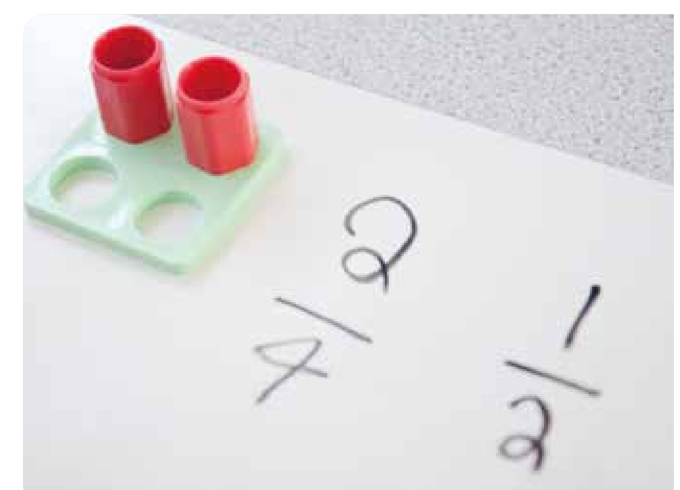
Typically, fractions arise for young children in measuring situations (which can include 'sharing'). Measuring is always approximate and, for this reason, children quickly need to share parts of units to describe amounts more precisely. The moral imperative for 'fair' shares usually draws children easily to the view that fractions are (and indeed should be) seen as equal parts of a whole.

The two main ways in which children will have experienced fractions initially in the Numicon programme are as descriptors and as operators (or as 'adjectives' and as 'verbs').

As a consequence of some measuring tasks, children may have met fractions as descriptors (adjectives), e.g. '26 and a half xs', or as the description of a distance, e.g. 'halfway' between 26 and 27 on a measuring scale.

On the other hand, the invitation to 'find half of 26' is an invitation to halve 26, with this verb introducing children to fraction ideas as processes. In Number, Pattern and Calculating 2, children will continue to work on fractions as operators in their instructions to 'halve' and 'quarter' amounts and sets of objects in ratio situations (see **Dividing**).

Importantly, in Number, Pattern and Calculating 2, children are now introduced to the idea of fractions as numbers with the activity of counting on and back in halves and quarters along a number line.



Dr Tony Wing – the theory behind Numicon: what we have learned in our work so far

"Teachers using Numicon quickly find themselves learning from children's responses – as do I and the rest of the Numicon authors.

Numicon is a continually growing understanding of the ways in which structured materials and imagery can be helpful to children in their learning of mathematics.

Using Numicon effectively involves understanding something of the theory behind Numicon, including an understanding of what young children face as they learn how to do mathematics.

The following section sets out what I and the other Numicon authors have learned in our work so far, in order to help with the use of the teaching materials."

Doing mathematics – being active and exploring relationships

It is most helpful to see mathematics as an activity, as something people 'do', actively, rather than as a lot of facts and techniques that have to be passively acquired.

The reason mathematics is thought so important for children in school is that we all want them, after they have left school, not just to partly remember those facts and techniques that they used to pass their exams, but to be able to do mathematics successfully when they later meet new and unfamiliar mathematical challenges in their everyday lives and in their work.

Being able to do mathematics involves being able to pick out key relationships in a situation and then manipulating those relationships to predict outcomes that we are interested in.

For example, in a practical shopping situation, key relationships could include those between prices, totals, budgets, currency structures and cash availability. We could manipulate all these to predict whether we can afford to buy something and, if so, how we could pay. Usually, such a situation would require us to do some calculating along the way. Here, the key relationships would be between numbers themselves: we might be manipulating number relationships by adding, or by finding a difference, to predict a total or an amount of change.

It is worth noting three aspects of doing mathematics in the shopping example.

- Firstly, there is the business of working out which quantities are important to us in the given situation, and how they relate to each other.
- Secondly, there is some calculating to do, with pure numbers.
- Thirdly, there is some interpreting of calculation results in order to predict what will happen when we decide what to do in the practical situation.

Identifying and manipulating key relationships in a situation in order to predict outcomes in this way is often called 'mathematical problem solving', and doing this essentially involves **exploring relationships** (the connections between things) within a situation.

Even a problem as simple as finding out, 'How many children are having lunch today?' involves using some kind of order relationship if we are to predict this successfully. For example, not counting the children in order may mean some are missed or some counted twice.

Numicon constantly encourages children to explore the relationships in situations, to see patterns and regularities, and to use these to make predictions. All this lies at the heart of doing mathematics at any level.

It is worth noting that when working on a real-life problem, such as calculating the cost of shopping, we tend to reach a point where we temporarily forget about the practical context we are in and just work with pure 'numbers'. This challenge of moving backwards and forwards between particular practical situations and an abstract world of numbers presents some of the most significant challenges that children face in learning how to do mathematics.

Of course, some problems crop up just within an abstract world of numbers, for example, as we learn how to calculate more effectively. Yet even these situations require us to **be active** and to **explore** and use the various relationships involved between numbers themselves.

Importantly, doing mathematics in our everyday lives and at work involves working out what to do in situations simply as they crop up; there is no helpfully arranged programme to life (as there is in school), and as adults we have to be able to cope with whatever comes up, in whatever order it appears. Children also have to learn to rise to the challenge of being able to do mathematics in new and often unfamiliar situations, not just try to remember selected techniques that are likely to come up at the end of a period studying a particular topic.

This has important implications for both our teaching and for our assessing. Children need to learn how to do mathematics in new and unfamiliar situations, how to actively explore relationships and to manipulate relationships between things in the same ways that mathematicians cope with new situations. Children – in their worlds – are learning to join in with the activity of doing mathematics in fresh fields.[1]

Generalizing, thinking and communicating

Doing mathematics makes the everyday world predictable in a surprising number of ways. When we board an aircraft, we expect to arrive safely and at our chosen destination; we expect fresh food to be available to us in our local shops as and when we want it; we expect electricity to flow in our homes and in our workplaces whenever we flick on a switch. How is it that these everyday expectations are met so often in most of our lives? The answer is that aeronautical engineers, navigators, logistics experts, electrical engineers and statisticians predict these things for us by doing mathematics.

Crucially, doing mathematics involves a unique way of **thinking** and **communicating** about situations; a special way of communicating that has been developing especially for the purpose of doing mathematics ever since humans first concerned themselves with quantities and relationships.

Interestingly, our mathematical communicating doesn't just happen between us and other people; we also constantly communicate mathematically with ourselves whenever we do mathematics. We call this thinking. Just try multiplying 481 by 37 and listen to the voice in your head as you work it out; you can hear your own thinking as you communicate your calculating with yourself.

It is important to understand how our thinking and our communicating develop. Building on Vygotsky's work, Sfard[2] argues that our thinking develops as our own 'individualized' version of the communicating that we do with others. The important implication is that children's mathematical thinking is their own individual version of the mathematical communicating they do with their teachers. Learning to join in with the mathematical communicating we use around them is how children learn to think and communicate mathematically for, and with, themselves.

1 For more on this view see Freudenthal, H. (1973) Mathematics as an Educational Task. Dordrecht: D. Reidel Publishing Company

2 Sfard, A. (2008) Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing. New York: Cambridge University Press

Making our lives predictable with mathematics and the ways that we communicate mathematically are closely connected. In fact, it is because mathematics aims to predict through seeing patterns and regularities in relationships that mathematical thinking and communicating has developed in the distinctive ways that it has. More specifically, mathematics makes situations predictable through **generalizing**. As a result, what we communicate with and about most often when doing mathematics are **generalizations**.

Even the number we call '3' is a generalization; we want children to understand it as meaning '3 of anything'. The number facts children are expected to remember are all generalizations. We want children to understand that '6 + 2 = 8' means '6 of anything and 2 of anything will together always make 8 things – whatever they are'. In geometry, when we talk about the angles of 'a triangle' adding up to 180°, we mean any triangle, not just a particular one we might have drawn in front of us.

Generalizations are important because they can be used to predict outcomes in particular situations. For example, once we've made the generalization that '4 x 25 = 100', we can predict that: the perimeter of a square of side 25 cm will be 100 cm; if we're given £25 a week for four weeks, we'll have been given £100 in all. We could even use it to calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

In other words, lots of different kinds of particular situations all become much more manageable because of that one generalization.

It is important for children to learn 'number facts' such as '6 x 3 = 18', but the most important thing (if such 'facts' are to be useful) is for children to reach them actively through generalizing themselves, and for children to learn how they can use these generalizations in their mathematical thinking and communicating to make the world they live in more predictable.

If 'generalizing' sounds like quite an abstract and difficult thing to do, try thinking of it as simply looking for patterns. The good news is that human beings are all very good at looking for and finding patterns in our experiences. Young children, in particular, have been phenomenally good at this since the day they were born; they learned to speak their mother tongues simply by being extraordinarily attentive to the patterns in the sounds around them – an incredible achievement.

Numicon taps into this incredible facility children have for spotting patterns in situations – for generalizing – wherever possible. This is at the heart of thinking and communicating mathematically.

How do we communicate mathematically?

Communicating with the sheer density of generalizations that we use when we are doing mathematics is not easy.

First of all, this is because generalizations distance us from the close particulars of our individual lives. This is one big reason why so many people find doing mathematics abstract and remote and – mistakenly – feel it is unconnected to their particular everyday world.

Secondly, it is very difficult to talk, and to think, about anything in general without imagining something in particular. When we want to say something about people in general, we usually have experience with particular people in mind (we only ever meet or hear about particular people); when we want to talk with children about triangles in general, we usually show them one triangle (or perhaps a few) in particular. So it is with number generalizations.

When we begin to talk about the generalization '3' with children, we often show them three things in particular (perhaps three counters), even though we want them somehow to interpret those particular three things as representing 3 of anything. 'Seeing the general in the particular'[3] is at the heart of doing mathematics.

The talking that we do in doing mathematics, both with ourselves and with others, is crucial to our developing mathematical thinking and communicating.

Over the centuries, we have developed sophisticated and effective ways of thinking and talking about numbers and

3 Mason, J. & Pimm, D. (1984) Generic Examples: Seeing the General in the Particular, Educational Studies in Mathematics, 15(3) p277-290

other generalizations in mathematics. Not surprisingly, children can sometimes have trouble joining in with them immediately.

Firstly, in our developed mathematical communicating about amounts and numbers of things in general, we somehow manage to turn our generalizations into objects – mathematical objects. For instance, since it would be very clumsy and awkward to be forever talking about our number generalizations as '6 of anything' and '242 of anything', in practice, we have got used to using a kind of linguistic shorthand and talking about just '6' and '242'. And this has consequences.

In this almost accidental way, we have opted to use number words as nouns and thus to speak and think about the generalizations '6 of anything' and '242 of anything' in the same way as we would if they were actually material objects in the world, like chairs, tables or frogs. We call these generalizations-we-make-into-things, these mathematical objects, numbers. In the previous example, we make the mathematical objects '6' and '242' simply by shortening our generalizing phrases into names.

It is important to realize that by naming something (simply by using a word as a noun), we implicitly announce that what we are talking about is a 'thing', an object; thus just by starting to use number words and symbols as nouns instead of adjectives in our language, we announce to children that in talking about numbers we are talking about things of some kind.

It is also important to remember that, in using number words as nouns, all we do is 'invent' numbers within our use of mathematical language – we don't actually make anything that is real in a world beyond language. A generalization (such as a number) is not a material thing; it is a thought formed in a way of using words (or symbols), and we start talking about such odd things very early in our work with children.

More curiously, most of us who have long ago learned to 'handle' numbers tend to feel as if we literally move them around and connect them as we do calculations, either on paper or 'in our heads' – again as if numbers had the properties of physical things. As numerate adults, we commonly calculate with symbols on a page, or on a calculator display, as if the numerals were somehow the abstract number ideas themselves.

As an example, try dividing 273 by 46 and see if you don't feel as if you are treating the numerals involved as if they are number objects of some kind, and that you just move them around and exchange some for others according to the particular rules you have learned. Don't numbers become just 'things on paper' for you, or numerals in your head or on a calculator display, as you work?[4]

As experts, we have come to think and talk about the generalizations we call numbers as things – as objects – as if these mathematical objects were the same kind of things as physical objects that we meet in our everyday worlds. We handle them, we move them around and we line them up on a page. In general, as we do this, we tend to use numerals as if they were the number 'things' that we have invented – simply, remember, by changing how we use number words. How is it possible for children to make sense of all these invisible mathematical 'objects' that we suddenly start talking about in association with numerals in their schooling?

The first thing to observe is that too many children don't ever really make sense of the number objects they meet in their schooling. In practice, we often just expect very young children to move smoothly from talking about 'three sweets' or 'three pencils' (using number words as adjectives referring to physical objects) to talking about just '3' (the same word now used as a noun without any accompanying referents) – as we do.

A little later on, as we introduce fractions, we expect children to move from talking about 'half a pizza', or 'half a bar of chocolate', to talking about just ½ as a thing that is not 'a half of' anything in particular.

This must all seem very strange, and most young children do not so much understand what we are doing as simply just 'try to go along with it'.

4 Significantly, most of us use imagery of various kinds as well as numerals, as we calculate. But when asked to imagine the number 'ninety four', for example, most of us picture two numerals ('94') – in that order – regardless of whether there is also an image such as a number line involved as well.

What do we show children as we talk about 'numbers'?

Confusingly for children, at the same point as we start talking about these generalizations (i.e. numbers), using nouns for them as if they were material things, we also commonly start focusing heavily on recording with numerals – as if, by coincidence, the written marks children can actually see on the board or the page are the numbers we are talking about.

There's a good reason for the move to symbols in mathematical communicating at this point: we can only talk about generalizations with what Bruner called symbolic representation[5]. We cannot draw a picture to show '6 of anything' because as soon as we make a picture, or count out 6 physical objects, we are showing children '6 of something'. It is symbolic representation (i.e. words and numerals) in particular that allows us to communicate most readily about invented, non-material things – in this case, pure numbers.

However, unless we are careful, children will quite reasonably – and commonly do (unconsciously) – think that the visible numerals we are showing them actually are the mysterious number things that we have now begun talking about.

The English language is not very helpful either, since, in English, we call numerals 'numbers' as well. We talk about 'the number of that bus' or 'the number of your house' when we are referring to numerals that we can see. In class, we ask children to 'write down' numbers while expecting them to draw numerals.

Importantly, at other times we also want children to understand that '3 and 3 are equal to 6'. What sense can that make to a child who thinks that the numerals '3' and '6' they are looking at are the numbers the teacher is talking about? It is very hard to make sense of such number relationships if all we have to communicate and to think with are number words and symbols; '3' and '3' together don't look like '6'– they look more like '33'.

It is our challenge to find ways of communicating about mathematical objects children can't see (pure numbers) in ways that avoid confusing them with numerals, and which also allow us all to explore relationships between these invisible things.

One key solution is to bring real physical objects and imagery into our communicating; to import special ways of **illustrating**, enactively and visually, relationships between the mystical generalizations we are talking about.

It is for this reason that, when introducing children to numbers, we commonly prepare for and supplement their use of numerals with a range of physical objects and imagery, using actions with objects and visual illustrations to help children 'see' and 'feel' how the invisible number objects we invent with our words relate to each other[6].

Crucially, this is the point at which we need children to start 'seeing the general' in the particular enactive and visual illustrations that we offer. We can help children a great deal with this in the discussions that we have with them as they work. If we use counters, or cubes, or beads, or beans

5 See Bruner, J. (1966) Towards a Theory of Instruction. Cambridge, MA: Harvard University Press

6 In this, we are combining Bruner's three modes of representation - enactive, iconic and symbolic – together in order to enrich children's learning as much as possible.

to talk about numbers, we want children to focus just on 'how many' there are, and to ignore the kinds of physical objects we are using. We need children to stress how many discrete objects are before them, and to ignore what sort of objects they are. 'Stressing and ignoring'[7] lies at the root of 'seeing the general in the particular' illustration, and our conversations with children as we 'illustrate' are crucial. We need to keep asking, 'Would it make any difference to our calculation if those counters were beans? Or what if they were pencils?'

In Bruner's terms, since by thinking and communicating mathematically we are working with generalizations (this is how mathematics helps us to predict), we will eventually do this most effectively by using symbolic representation, e.g. by using numerals and words in our writing and talking to represent our number generalizations.

However, numeral symbols and words are merely conventionally agreed marks on a page and sounds that we hear and – crucially – are dependent for their interpretation upon the prior and accompanying experiences of both action and imagery that led up to the generalizing they symbolize.

7 For helpful discussion on this and related views see Mason, J & Johnston-Wilder, S. (Eds.) (2004) Fundamental Constructs in Mathematics Education. London: Routledge Falmer. P126ff

Thus, effective use of numerals by children (symbolic representation) in their thinking and communicating – in other words, the calculating we want them to be able to do – is dependent upon their prior and accompanying use of enactive and iconic representation in their experiences with numbers of things.

Children's learning to think and communicate mathematically with generalizations will eventually lead them to mastery of associated symbolic representation (numerals and words). In order to reach and sustain that mastery however, children's route lies necessarily through use of enactive and iconic representation (action and imagery). The most effective teaching therefore involves children 'individualizing' the actions, imagery, words and symbols we use, and joining us in using them in the mathematical communicating of their classrooms.

It is also important to remember that most effective mathematical thinking and communicating at all levels involves a rich blend of enactive, iconic, and symbolic representation together. Action and imagery always support the interpretation of mathematical symbols and there is no point at which children should be expected to leave actions and imagery (physical illustration) behind to do 'grown up' thinking. As children's thinking develops, they 'internalize' the actions and imagery that have led to their effective use of symbols, but physical materials and imagery should always be available in classrooms for children to call upon as new ideas are met, and familiar ones reviewed.

To sum up this part of the theory behind Numicon, numerals (and words) are vitally important symbols that gradually become ciphers for the generalized number objects used in advanced mathematical communicating and thinking.

However, as arbitrary conventional symbols, numerals and words cannot illustrate any number relationships. If children are to learn how to handle number generalizations and their connections effectively, they need to have ways of mediating their mathematical communicating (and thus mathematical thinking) with illustrations that will help children 'see the general in the particular'.

Sfard (Op. cit.) calls the objects and images used for this illustrative purpose communication mediators, since they are used to mediate communicating with children. Such objects and imagery subsequently come to mediate children's mathematical communicating with themselves – their thinking about numbers.

Numicon introduces children to thinking and communicating about numbers with a combination of numerals and words in writing and talking, and by mediating this symbolic communicating with physical objects and imagery (enactive and iconic communicating) to illustrate the generalizing that both invents 'numbers' and establishes the relationships between them.

Which communication mediators should we use? Does it matter?

As noted previously, doing mathematics centres around generalizing and using generalizations. In learning how to do mathematics, we become capable of solving an increasing variety of mathematical problems as we call upon an increasing range of generalizations from our past experiences.

Young children usually first learn to generalize about quantities and amounts and talk about number generalizations in their communicating with us, and they then call upon these early generalizations when they later need to study, and to calculate with, relationships between quantities and amounts in ever more complex situations.

Children's facility in 'handling' the generalizations we call numbers – their ability to calculate with whole, positive numbers, fractions, negative numbers, for example – is fundamental to almost all of their subsequent progress with mathematics.

We also note that working with numbers – calculating – takes children into a world of invented objects that, although not real, are very significantly connected. Unless children can also begin to generalize about number connections – to generalize about their number generalizations – their calculating will remain very primitive. Typically, children who fail to make much progress with calculating remain restricted to the use of laborious and very basic counting procedures. Fortunately, the regular and systematic relationships between the numbers we invent both makes generalizing about them possible, and also gives us a clue as to which illustrations are most likely to be helpful to children exploring them.

Numbers are invented in well-organized systems, and our illustrating needs to reflect their systemic relationships.

In essence, in order to calculate effectively, children need to explore the relationships of numbers to each other. In other words, they need to explore the various ways in which the generalizations of our mathematical communicating about quantities connect with each other. For instance, the ways in which numbers are ordered is important, as are equivalences such as '6 + 2' being equivalent to '8'. The fact that it doesn't matter which way round we add two numbers together, they are equivalent to the same number total is important. As is the fact that if we 'add' more than two numbers together it doesn't matter what order we do that in. The numbers '0' and '1' seem to be peculiar in that 'adding 0' and 'multiplying by 1' seem to have no effect at all. Adding seems to be the opposite of subtracting, while multiplying seems to be the opposite of dividing. Interestingly, adding and multiplying seem to be closely connected with each other, as do dividing and subtracting. These are all generalizations about how 'numbers' relate to each other, and they are crucial to the effectiveness of children's calculating.

Scattered (unstructured), random collections of loose objects as illustrations render number relationships, such as those outlined, obscure and are only really useful as opportunities for children to impose relationships upon. Such unorganized collections of, for example, cubes and counters, initially offer opportunities for early counting practice. However, if, as illustrations, they remain unorganized, they are a poor foundation for calculating. It is very difficult to mediate any communicating about number relationships with unorganized collections.

Numicon uses objects and imagery specifically to mediate communicating about number relationships. Numicon brings physical objects and imagery into mathematical communicating that illustrate, above all else, the ways in which numbers are connected. When random collections of discrete objects are introduced in Numicon, children are always expected to put order upon them: to make relationships in what they see.

Numicon also uses a variety of actions, physical objects and imagery because children need to be generalizing from their experiences and conversations and children can only generalize from a variety of experiences.

Number lines of various kinds are used to foreground the order relationships of numbers and Numicon Shapes to give regularity, pattern and two-dimensional shape to number relationships.

Number rods are used to allow children to relate the sizes of numbers to each other in many more ways than are possible with number lines.

Loosely arranged collections of objects are only introduced to offer children important opportunities to impose their own relationships upon such situations: both order structures (when they count) and shape regularities (when they 'find how many' without counting).

By choosing a range of physical materials and imagery suited particularly to illustrating relationships, Numicon offers children the crucial enactive and iconic experiences that enable them to manipulate symbolic representations of their generalizations (numerals and words) effectively in their calculating.

The importance of context

We began this discussion by noting that we want children to leave school able to do mathematics as required in their everyday lives and in their work. In other words, we want children to be able to solve new and unusual mathematical problems when they meet them. Children thus need to be able to explore and pick out key relationships in new and unfamiliar situations, and manipulate them in order to render those situations predictable.

Usually, doing mathematics also involves children moving into the abstract world of number generalizations at key stages – pure calculating – before returning to the practical situation with one or more numbers which need interpreting in the particular context of the problem.

Numicon teaching materials advocate approaching children knowing when the generalized facts and techniques of mathematical communicating are useful through contexts and talking. Within Numicon, each group of activities begins with a carefully chosen particular context, in which the mathematics that is to be learned would be found useful.

Work begins on an activity group by talking about the relationships within which a need for some kind of mathematical response is established; children discuss their initial responses to the questions and challenges involved before moving to the generalizing mathematics to be learned through those activities.

In associated practice activities, children have opportunities to use the general mathematics they are learning in further particular contexts, thus learning from further opportunities to judge when such mathematical generalizing can help.

In Explorer Progress Books tasks, children are presented with challenges that invite them to use mathematics they have been learning, but in unusual and unfamiliar contexts. As their name implies, these tasks invite children to explore relationships in fresh fields.

Summarizing

The key to understanding Numicon is to recognize that doing mathematics involves learning how to **communicate mathematically**, and that mathematical communication is essentially about **generalizations**: it is through generalizing in mathematics that we make our particular worlds predictable.

In the process of doing mathematics, we often make our generalizations into things that we cannot see: *mathematical objects.*

Communicating is at the heart of doing mathematics because that is where our *mathematical objects* – our generalizations – are made.

However, communicating with generalizations is not easy; to help children move between the worlds of mathematical generalizations and of particular situations, we will always need to illustrate our communicating by **being active**, by **illustrating**, and by **talking** as children **explore relationships** they can physically see and feel.

Glossary

Most mathematical terms used in the *Number, Pattern and Calculating 2 Implementation Guide* and the Teaching Resource Handbook can be found in a good mathematics dictionary such as the *Oxford Primary Maths Dictionary.*

Other terms you might not be familiar with are explained in this glossary.

base-ten apparatus

A set of concrete materials, systematically designed to help children understand our place value system. Small cubes, sticks of ten cubes, flat squares of 100 cubes, and large cubes of 1000 small cubes are used when talking about ones, tens, hundreds and thousands respectively. (See Fig. 1.)

bead string

A string of coloured beads (usually red and white, but not necessarily) arranged so that successive decades of beads are alternately coloured (see Fig. 2). There is enough free space available on the string for beads to be moved backwards and forwards according to the numbers and calculation conversations being mediated by use of the bead string.

bridging across a multiple of 10 when adding or subtracting

Bridging is a calculating technique that involves partitioning (splitting) the number to be added or subtracted. Bridging can be used across any number, but bridging across multiples of 10 is especially useful, since this exploits the very basic adding and subtracting facts to 10 that children learn early on, e.g. 8 + 9 = (8 + 2) + 7 = 17 (see Fig. 3).

Bruner

Jerome Bruner (1915–2016) was an extremely distinguished and influential psychologist. His distinctions between enactive, iconic and symbolic representation have been particularly influential in the field of education.

column value

Numbers are often arranged in columns, with each column having a place value, e.g. hundreds, tens or ones. The numeral '2' in '327' is said to have a column value of two tens.

(See also **quantity value**.)

communication mediator

A communication mediator is an object, image or action that helps communication in some way. These communication mediators need to be carefully introduced to children, e.g. Numicon Shapes and number rods become communication mediators when they help to illustrate discussions about the relationships between numbers; number lines, graphs, bead strings can all act as communication mediators. However, these things only become communication mediators if they are used to support communication. Any physical object or image is just a physical object or image unless it is actually supporting communication; there is 'no magic in the plastic'.

enactive, iconic and symbolic representation

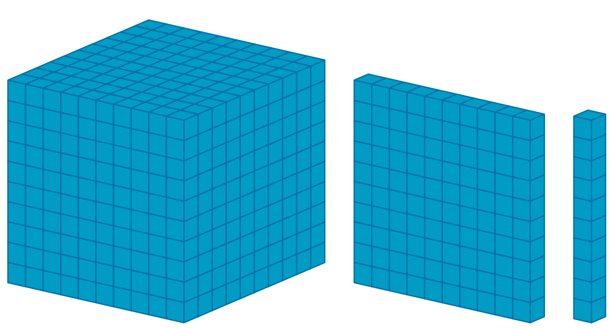
Jerome Bruner (1966) distinguished three key ways in which we humans represent experience to ourselves: through enactive (internalized action), iconic (sensory) and symbolic (forms of language) representations. In the Numicon approach we seek to combine all three forms of representation so that children experience number ideas through action, imagery and conversation.

enumerate

To name how many distinct objects there are in a collection. In Numicon, this term is used in activities that focus on finding how many without counting each individual object; children do this instead by making Numicon Shape patterns.

generalization

A statement or observation (not necessarily true) about a whole class of objects, situations, or phenomena. Generalizations are essential and everywhere in mathematics, and for this reason children need to generalize, and to work with generalizations, constantly. Numbers are generalizations, as are rules about numbers, such as, 'it doesn't matter which way around you add two numbers, you will always get the same answer'.



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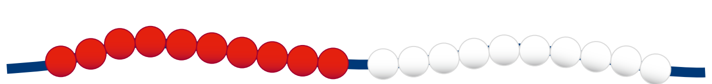

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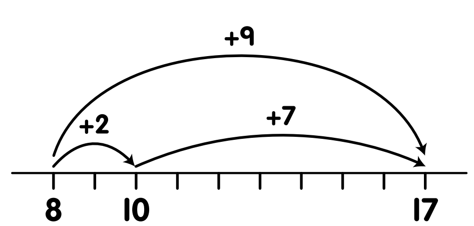

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number fact

The term 'number fact' usually refers to an operation on two or more numbers, together with its outcome. So '6 + 3 = 9' is a number 'fact', as is '256 ÷ 16 = 16'. In UK schools, these are commonly referred to as 'number bonds'.

number names/objects/words

Adults commonly talk about numbers as if they are objects, i.e. we often use number words such as 'four', or 'twenty-three', as nouns; we ask children questions such as, 'What is seven and three?' In our language, nouns name objects, so we all commonly (and unconsciously) assume that if we use a word as a noun it must be naming an object. So when we use number words as nouns we assume they must be being used to name number objects – thus, according to the way we use words, numbers are often treated as if they are objects.

It is important to remember that we do not always use number words as nouns; quite often we use those same words as adjectives, as in, 'Can you get me three spoons?' One of the key puzzles for children to work out is how (and when) to use number words as adjectives and when as nouns.

number sentence

The metaphor of sentence (from the use of the word 'sentence' in literacy and grammar) is sometimes used to refer to the writing of a number fact in horizontal form, from left to right. So, 4 + 23 = 27 is a number sentence because it is written in the same graphical manner as a normal written sentence in prose.

number trio

A number trio describes a set of three numbers that relate inverse adding and subtracting facts, e.g. 3, 4 and 7 (see Fig. 4). These are used in Numicon, together with specific forms of illustration, to support children's development of adding and subtracting number facts.

numerals

Numerals function as shorthand for number words. The numeral '5' is shorthand for the word 'five'. When used as a noun, the numeral '5' is often said to represent or stand for the number object we call 'five'.

Numicon Shape pattern

When we refer to a Numicon Shape pattern we are referring to the system of arranging objects or images (up to ten in number) in pairs alongside each other that is sometimes called 'the pair-wise tens frame'. Fig. 5 shows the Numicon 7-pattern.

Numicon Shape

Numicon Shapes are pieces of coloured plastic with holes (ranging from 1 hole to 10 holes) arranged in the pattern of a pair-wise tens frame (see Fig. 6).

Piaget

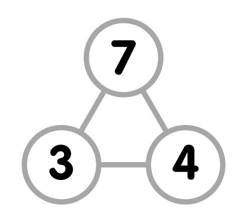
Jean Piaget (1896–1980), was a philosopher and psychologist who spent years studying young children and how they learn. He believed that the development of knowledge is based on developing cognitive structures and that children should be actively involved in their own learning.

quantity value

The numeral '2' in '327' is said to have a quantity value of 20 (twenty). (See also **column value**.)

Vygotsky

Lev Vygotsky (1896–1934) contributed a uniquely social dimension to the study of children's thinking. In particular he stressed the role of expert adults in supporting a child's new learning; this occurs optimally in a child's 'zone of proximal development'. Importantly, he saw the development of a child's thinking as crucially influenced by what he characterized as the 'internalization' of speech.



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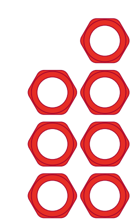

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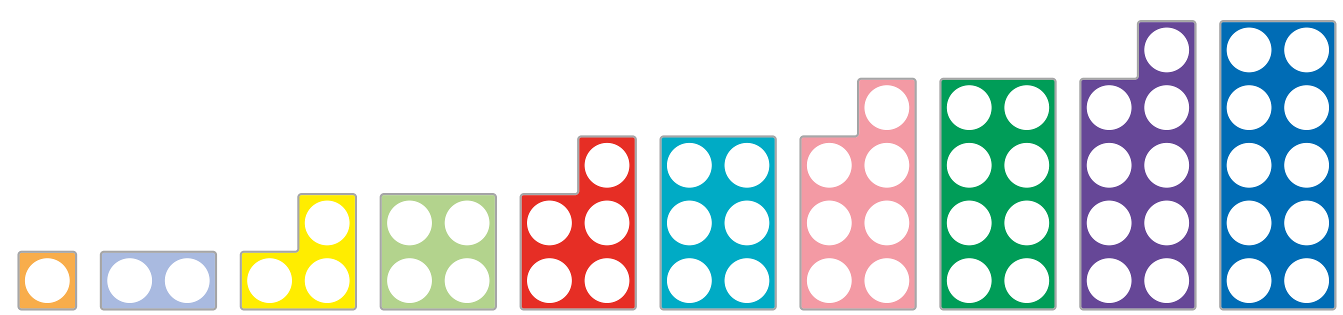

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numicon

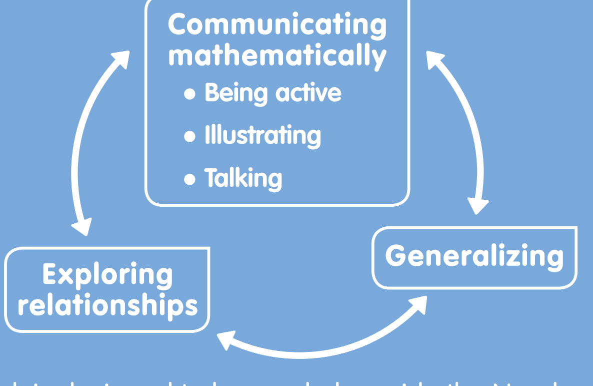
Implementation Guide 2

Number, Pattern and Calculating

Numicon is a proven approach for teaching and learning maths that builds deep understanding and engagement. Through active investigation with problem-solving at its heart and supported by structured apparatus, children reason and communicate mathematically with confidence.

Your **Implementation Guide** contains:

- Practical guidance on how to get started with the Numicon resources
- An introduction to Numicon as a distinctive approach to children's mathematical learning
- Guidance on how to plan and assess effectively, and create a number-rich environment in the classroom
- Key mathematical ideas to support teachers' subject knowledge
- The theory behind Numicon from Dr Tony Wing



This book is designed to be used alongside the Number, Pattern and Calculating 2 Teaching Resource Handbook, Explorer Progress Books, Explore More Copymasters and the online Planning and Assessment Support.

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