numicon®

Number, Pattern and Calculating 5

Implementation Guide

Written and developed by

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www.oxfordprimary.co.uk/numicon

About Numicon

Numicon is a distinctive multi-sensory approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Numicon was founded in the daily experience of intelligent children having real difficulty with maths, the frequent underestimation of the complexity of the ideas that young children are asked to face when doing maths and recognition of the importance of maths to them and to society as a whole.

Numicon aims to facilitate children's understanding and enjoyment of maths by using structured imagery that plays to children's strong sense of pattern. This is done through research-based, multi-sensory teaching activities.

Numicon takes into account the complexity of abstract number ideas and seeks to foster the self-belief necessary to achieve in the face of challenge or difficulty.

Through the combination of communicating mathematically (being active, talking and illustrating), exploring relationships and generalizing, children are given the support to structure their experiences: a vital skill for both their mathematical and their overall development.

A multi-sensory approach, particularly one that makes use of structured imagery, provides learners with the opportunity to play to their strengths, thereby releasing their potential to enjoy, understand and achieve in maths. By watching and listening to what children do and say, this enjoyment in achievement is also shared by teachers and parents.

Numicon strives to support teachers' subject knowledge and pedagogy by providing teaching materials, Professional Development and on-going support that will help develop a better understanding of how to encourage all learners in the vital early stages of their own mathematical journey.

Contents

Welcome to Number, Pattern and Calculating 5

Before you start teaching, take some time to familiarize yourself with the Number, Pattern and Calculating 5 starter apparatus pack, the teaching resources and the pupil materials, to help you and the children get the most out of using Numicon.

Use this Implementation Guide:

- to find out more about what Numicon is
- to find out how using Numicon might affect your mathematics teaching
- to learn about the key mathematical ideas children face in the activity groups
- for more detailed information on the theory behind Numicon from Dr Tony Wing.

You will find guidance on how to get the most out of teaching, planning and assessing using Numicon in the Numicon Planning and Assessment Support.

You will find more information, videos, ideas and free resources on the Numicon website: www.oxfordprimary.co.uk/numicon.
Here, you can sign up for our newsletter, which includes suggestions for topical mathematics and updates on Numicon.


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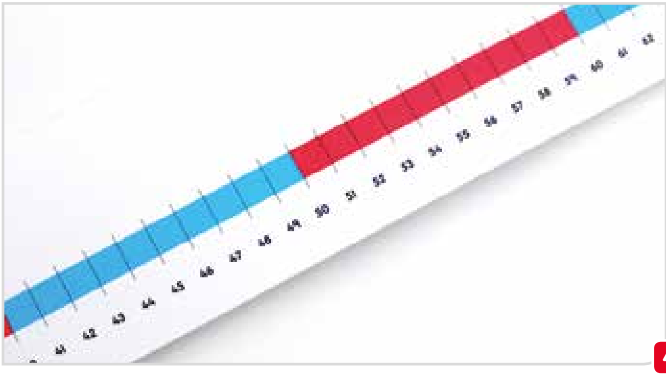
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What's in the Numicon starter apparatus pack?

The following list of apparatus supports the teaching of Number, Pattern and Calculating 5. These resources should be used in conjunction with the focus and independent activities described in the activity groups.

Starter apparatus pack contents

- Numicon Shapes – box of 80 (× 2)
- Numicon Coloured Counters – bag of 200 (× 2)
- Numicon Baseboard Laminate – set of 3 (× 2)
- Numicon 0–100 cm Number Line – set of 3 (× 2)
- Numicon 1 000 000 Display Frieze
- Numicon 0–1·01 Decimal Number Line
- Numicon $^{-}$12–12 Number Line
- Numicon 10s Number Line Laminate (× 4)
- Numicon Fraction Number Line Laminate
- Numicon Spinner (× 4)
- Numicon 0–100 Numeral Cards
- Numicon 1–100 Card Number Track (× 3)
- Number rods – large set
- Numicon 1–100 cm Number Rod Track (× 3)
- Extra Numicon 10-shapes – bag of 10 (× 3)
- Numicon Feely Bag (× 2)
- Magnetic strip

Numicon Shapes 1

These offer a tactile and visual illustration of number ideas. The Shapes are also a key feature of the *Numicon Software for the Interactive Whiteboard*, useful for whole-class teaching sessions. However, the Software is not a substitute for children actually handling the Shapes themselves. It is strongly recommended that children are provided with their own individual set of Numicon Shapes 1–10 for use in whole-class sessions.

Numicon Coloured Counters 2

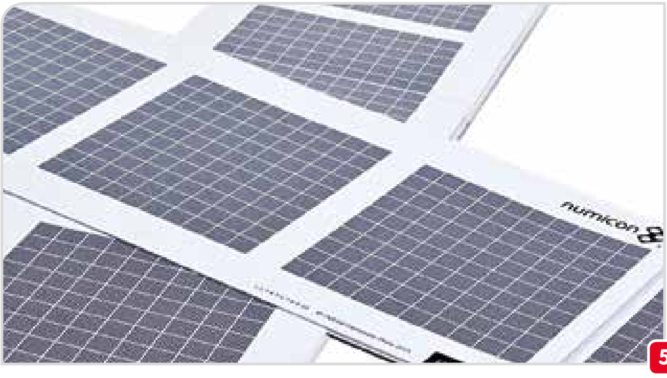
These red, yellow, blue and green Counters can be used for building arrays when multiplying and dividing, for arranging into Numicon Shape patterns in counting activities, and for exploring patterns and possibilities. Using them with the baseboard laminates allows children a clearly defined field of action upon which to create their arrays or patterns.

Numicon Baseboard Laminate 3

This double-sided laminated square baseboard is an empty 100 square, scaled to take Numicon Shapes and Counters. The white side is used in many activities, providing a defined 'field of action' for number and pattern investigations. The orange side is a decimal baseboard laminate which can be used to help children continue to explore decimal numbers up to two decimal places. It offers a possible representation of an expanded 1-shape for children to represent decimal parts of numbers on.

Numicon 0–100 cm Number Line 4

The points on this number line are 1 cm apart and are labelled from 0–100. The number line is divided into decade sections, distinguished alternately in red and blue, to help children find the 10s numbers that are such important signposts when children are looking for other numbers. This resource can also be used with number rods.



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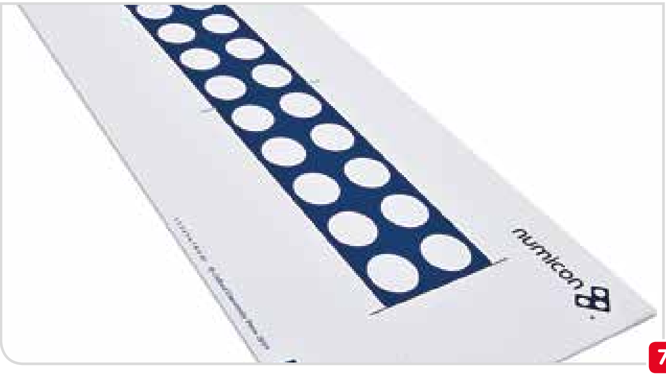

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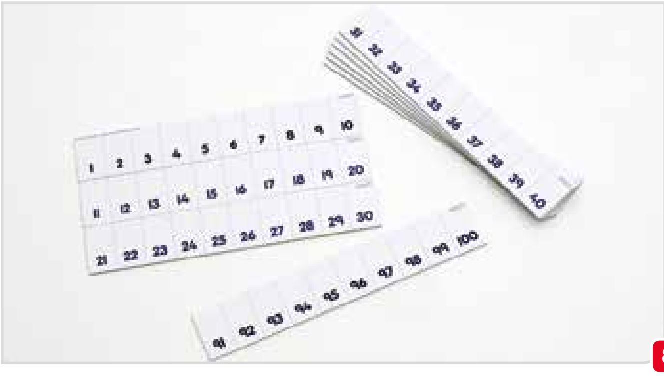

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Numicon 1 000 000 Display Frieze 5

This display frieze shows 100 blocks of 100 squares, each made up of 100 dots helping children recognize 1 000 000 as a cube number. The sections can be arranged end-to-end horizontally or as an array such as in a square of 10 × 10 blocks, helping children also recognize 1 000 000 as a square number. It provides a visual reference point for the scale of large numbers as well as supporting discussions about the application of square and cube numbers to area and volume.

Numicon 0–1·01 Decimal Number Line

The points on this number line represent thousandths, hundredths and tenths. Zero and one are labelled while the other points are left blank to encourage children to think about what each point shows. Children could record decimals in tenths, hundredths or thousandths on this number line.

Numicon $^{-}12$–12 Number Line 6

This number line shows negative and positive numerals. It provides a visual reference for counting and calculating with negative numbers and can be displayed on the wall or given to children for use on their tables.

Numicon 10s Number Line Laminate 7

This laminated number line, scaled to take Numicon Shapes, shows Numicon 10-shapes laid horizontally end-to-end with points marked, but not labelled. The points can be labelled with multiples, fractions, decimals, percentages or negative numbers, using a whiteboard pen, to support children with exploring number relationships and calculating.

Numicon Fraction Number Line Laminate

This laminated number line starts at zero and has fifty unlabelled points. The points can be labelled with any denomination of fraction children are working with. This provides a valuable tool for exploring equivalences and comparing fractions with different denominators.

Numicon Spinner

The Numicon Spinner can be used in many practice activities. Different overlays (provided as photocopy masters) can be placed on the spinner to generate a variety of instructions for children to follow, including: numerals, percentages and symbols of arithmetic notation. The spinner also features on the *Numicon Software for the Interactive Whiteboard.*

Numicon 0–100 Numeral Cards

This pack of 0–100 numeral cards can be used in several activities for generating numbers which children then work with. It is also used in some of the whole-class and independent practice activities and games.

Numicon 1–100 Card Number Track 8

This number track is divided into ten strips, numbered 1–10, 11–20, 21–30 and so on. The sections can be arranged horizontally end-to-end as a number track, or as an array similar to a 100-square.


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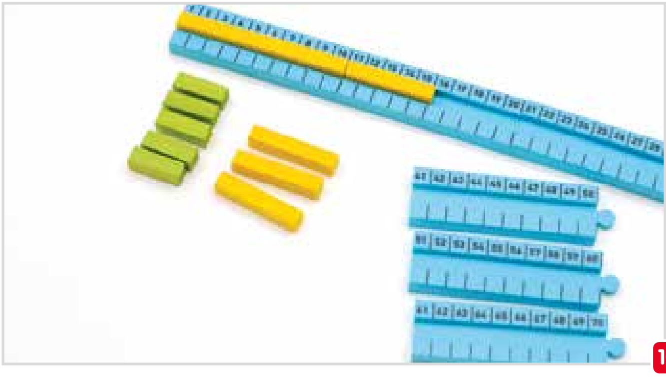
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Number rods 9

A box of number rods contains multiple sets of ten coloured rods, 1 cm square in cross section. The shortest is 1 cm in length, the longest 10 cm. These offer another structured illustration for number and are used alongside Numicon Shapes in many of the activities. Being centimetre-scaled, they can also be placed along the Numicon 0–100 cm Number Line.

Numicon 1–100 cm Number Rod Track 10

Use this for teaching about place value, partitioning, multiplying and dividing. The decade sections click together into a metre-long track. Designed to take number rods, it can be separated easily into sections and arranged as an array.

Numicon Feely Bag

Children use the Feely Bag when playing games such as 'What number rod is in the Feely Bag?' to continue to develop their understanding of number properties. They also use it to generate numbers for problems involving place value and money.

Magnetic strip

This self-adhesive magnetic strip can be cut into pieces and stuck onto the back of Numicon Shapes or number rods so that they can be used on a magnetic whiteboard.

Available separately

Numicon Software for the Interactive Whiteboard 11

This rich interactive tool is designed for use with the whole class to introduce key mathematical ideas. It includes: number lines featuring Numicon Shapes, the Numicon Pan Balance, objects for counting, coins, Numicon Spinners and much more.

Individual sets of Numicon Shapes 1–10

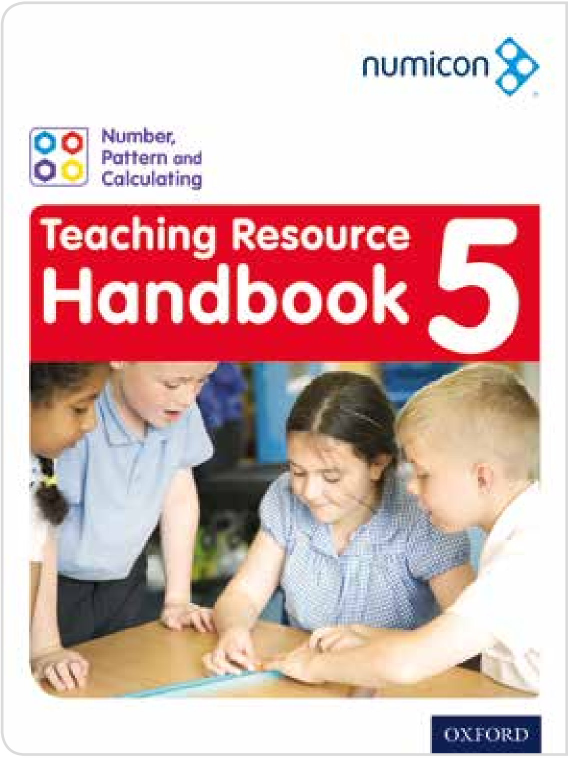
These are designed for multi-sensory whole-class lessons, where each child has their own set of Shapes and is encouraged to engage with them. They are especially useful when used in conjunction with the *Numicon Software for the Interactive Whiteboard* to help teachers assess children's individual responses from the Shapes children hold up.

Numicon Pan Balance 12

Using Numicon Shapes or number rods in this adjustable Pan Balance enables children to see equivalent combinations, helping them to understand that the '=' symbol means 'is of equal value', thus avoiding the misunderstanding that it is an instruction to do something. Children can easily see which Shapes are in the transparent pans. A virtual balance is also featured on the *Numicon Software for the Interactive Whiteboard*.

Other equipment

Some activities use apparatus found in most classrooms, e.g. **sorting equipment, base-ten apparatus** and **interlocking cubes** as well as real-life items such as receipts and clothing labels. Opportunities to use these are highlighted in the '**have ready**' sections of each focus and independent practice activity.



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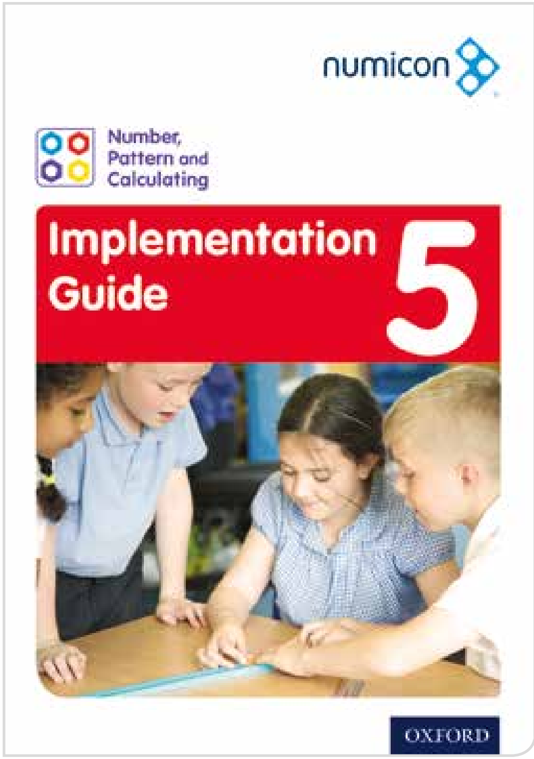

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What's in the Numicon teaching resources?

Number, Pattern and Calculating 5 Teaching Resource Handbook 13

This contains thirty activity groups clearly set out and supported by illustrations. Each activity group begins with the educational context, learning opportunities, assessment opportunities and important mathematical vocabulary that support mathematical communicating in the activity group. To support teachers' assessing of children, there are notes on what to 'look and listen for' as children work on the activities, as well as suggestions for whole-class and independent practice. Photocopy masters supporting the thirty activity groups are included at the back of the Teaching Resource Handbook.

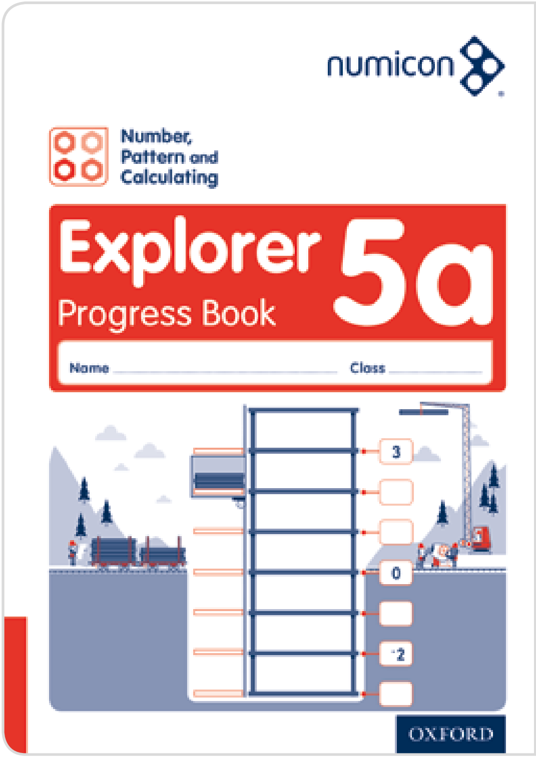
Support for planning and assessment is included at the front of the handbook. There you will find:

- information on how to use the Numicon teaching materials and the physical resources
- long- and medium-term planning charts that show the recommended progression through the thirty activity groups
- milestones to help assess how children are progressing in their learning
- an overview of the activity groups.

Number, Pattern and Calculating 5 Implementation Guide 14

This introduces what Numicon is and how Numicon helps children meet the demands of learning mathematics. It also includes some practical advice about what to do when preparing to teach with Numicon and answers some key questions about how to use Numicon in practice. The 'Key mathematical ideas' section provides useful explanations about the important mathematical ideas children will meet in the thirty activity groups of the Teaching Resource Handbook and how these ideas develop when teaching with Numicon. There is also a further chapter with more background detail on the research that inspired Numicon and the rationale behind the pedagogy.

The different sections of the Implementation Guide can be accessed as and when necessary to best help you with your teaching.



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Number, Pattern and Calculating 5 Explorer Progress Books 15

The Explorer Progress Books offer children the chance to try out the mathematics they have been learning in each activity group. In children's responses, teachers will be able to assess what kind of progress individual children are making with the central ideas involved in each activity group.

It should be stressed that the challenges in the Explorer Progress Books are not tests. There are no pass/fail criteria; the challenges are simply designed to reveal how well children can use the mathematics they have been learning in a new situation.

Being able to use mathematics in an unfamiliar situation is a significant indicator of children's understanding. Many of the tasks from the Explorer Progress Books set mathematics in a new or different context and, where possible, the challenges are set in an open way, inviting children to show how they can reason with the ideas involved rather than testing whether they have learned a routine solution to routine tasks.

As with children's classroom activity work, a range of apparatus should be freely available to children as they reason with the ideas in their Explorer Progress Books.

In addition, there is also scope for self-assessment in each Explorer Progress Book in the form of a Learning Log, which can be used flexibly throughout a term, or to summarize the learning of a block of work.

Each Explorer Progress Book covers ten activity groups, so three books are provided for the thirty Number, Pattern and Calculating 5 activity groups.

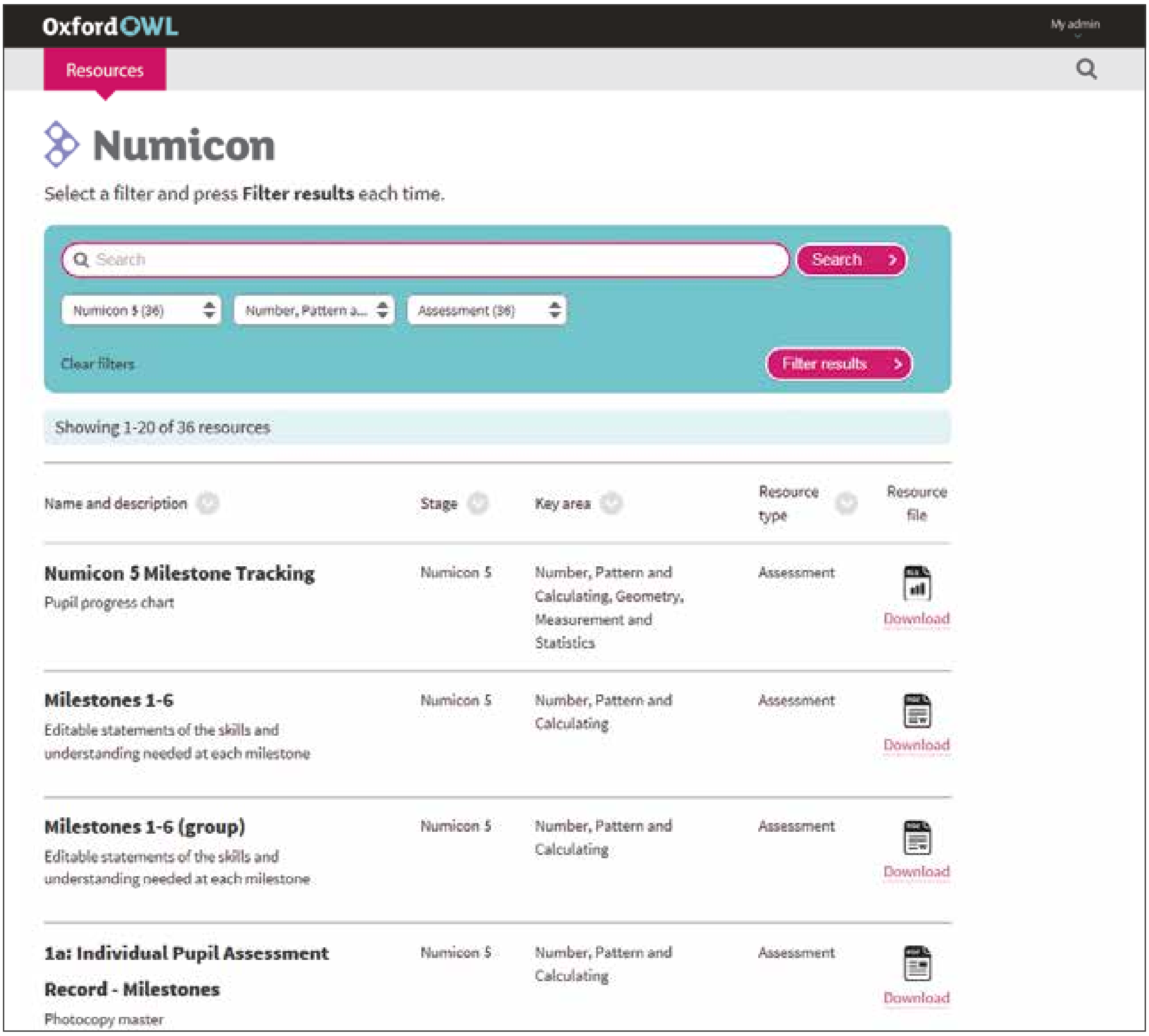
Number, Pattern and Calculating 5 Explore More Copymasters 16

The Explore More Copymasters offer children the chance to practise and discuss mathematics at home with a parent or carer.

An activity has been included for each activity group so that children have ongoing opportunities to practise their mathematics learning outside the classroom.

Each activity is supported by information for the parent or carer on the mathematics that has been learned in class beforehand, and the learning point of the activity itself. Guidance on how to complete the activity is included, as well as suggestions for how to make the activity more challenging or how to develop the activity further in a real-life situation.

The Explore More Copymasters can be given to an adult or child as a photocopied resource.



Numicon Planning and Assessment Support

The Numicon Planning and Assessment Support is available in the Teaching and Assessment Resources on the Oxford Owl website and contains a wealth of digital resources that will assist you with your planning and assessment needs.

Within the support, you will find a range of resources including short videos introducing Number, Pattern and Calculating 5 and offering advice that will help you get started teaching with Numicon.

There is also an editable summary of each activity group, including the title and number of the group, the educational context, the learning opportunities, the mathematical words and terms to be used with children as they work on the activities, and the assessment opportunities. There are suggestions for how to use these activity group summaries in your planning, as well as editable short-, medium- and long-term planning frames.

There is a milestone tracking chart to support you in monitoring each child's progress throughout the year. Assessment grids are also available to record children's achievements in each activity group and during work in the Explorer Progress Books.

Charts that map Numicon to the English, Welsh, Scottish and Northern Irish curricula have been included in these resources, as have charts showing the progression of the Numicon teaching programme across Number, Pattern and Calculating and Geometry, Measurement and Statistics.

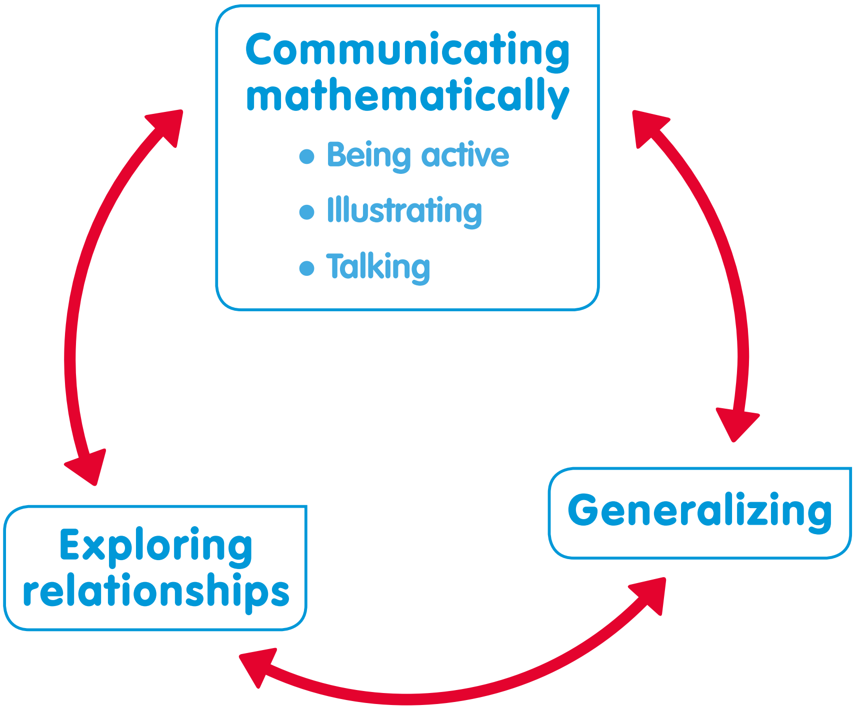
Printable versions of all the photocopy masters and resources for creating mathematics displays in your classroom are also provided.

What is Numicon?

In order to illustrate how using Numicon in your teaching can help children learn mathematics, this section looks at:

- what Numicon is
- what doing mathematics demands of children
- how using Numicon helps children meet these unique demands.

If you would like further information on the theory behind Numicon from Dr Tony Wing, please turn to page 68.



What is Numicon?

Numicon is a distinctive approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Communicating mathematically

Doing mathematics involves communicating and thinking mathematically – and these are two sides of the same coin. We think in the same ways that we communicate, and communicate in the same ways that we think. As children learn to communicate mathematically, they learn to think mathematically. This involves them in the following:

Being active: Teaching and learning with Numicon requires children to be active. This does not only mean being physically active (e.g. fitting physical objects together, finding a number on a number line, drawing a shape), but is a requirement reflecting the understanding that mathematics itself is activity: mathematics is something children are learning to *do*.

What this means in practice is that it is always the children themselves who are to do the mathematics. Telling children (or showing, or explaining) 'what to do' can encourage children to be passive. Numicon asks that the children do the mathematics (i.e. both the activity and the thinking). In other words, children actively learn to do mathematics for themselves.

Illustrating: Doing mathematics (i.e. thinking and communicating mathematically) necessarily involves illustrating, because mathematics is about studying relationships between objects, actions and measures, and it is impossible to explore such relationships without some kind of spatial imagery being involved. Even when thinking about just two numbers, one of them 'comes before' the other one on a number line or is said to be 'bigger' or 'higher'; these relations all involve spatial imagery.

Numicon explicitly involves illustrating in every activity. This is how relationships in a wide variety of contexts are both explored and communicated.

Talking: Since doing mathematics involves communicating mathematically (both with others and with ourselves), doing mathematics also involves talking. Talking is an essential aspect of all Numicon activity.

Throughout Numicon, talking means dialogue in which points of view are exchanged between teachers and children, and between children and children. All partners in dialogue will be actively involved, not just passively hearing or waiting to speak. Such exchanges encourage reflective thinking as children learn to discuss different points of view, both with others and with themselves.

Exploring relationships (in a variety of contexts)

Doing mathematics involves **exploring relationships** (i.e. the structure) in any situation in order to develop some kind of control in that area. Relationships can be between amounts, between positions, between shapes, between things that are varying, between things that are constant, or between any combination of all or any of these things. Fully understanding the relationships in a situation makes it predictable, thus making it possible to find out and/or manage what is needed. The mathematical reasoning children are asked to do is their expression of relationships they perceive.

Numicon ensures that children explore the relationships within a wide range of contexts so that they learn not only *how* to do mathematics, but *when* the mathematics they are learning is useful.

Generalizing

In doing mathematics, exploring relationships and looking for patterns in various situations lead to **generalizing**. It is generalizing that allows us to make whole ranges of new situations predictable.

Numbers are generalizations that we all use to make predictions when calculating. For example, the '6', '2' and '8' in the number sentence '6 + 2 = 8' are generalizations; 6 of anything and 2 of anything will together always make 8 things, whatever they are.

'The angles of a triangle add up to 180°' is a generalization that is often used when doing geometry; 'the area of a circle is πr^2' is another that is used when measuring.

In each of these cases, noticing patterns in relationships allows us to generalize about an infinite number of other, similar situations.

It is because we make and use generalizations continually as we do mathematics that mathematical thinking and communicating will appear to have an abstract character for children if they are not involved in doing the generalizing for themselves.

Communicating mathematically, **exploring relationships** and **generalizing** all come together when *doing* mathematics.

What doing mathematics demands of children

In learning to do mathematics at the age they are asked to, children face a unique challenge in their school curriculum: thinking and communicating about abstract objects.

Pure numbers, such as 6 and 254, are abstract objects and in no other subject are children asked to work with and reason about such things so early. It is not surprising that children can hesitate, be puzzled, get stuck, or take time to work things out for themselves.

Most mathematicians will say that doing mathematics is about looking for patterns in situations. It is, but there is more to it than that. When you find a pattern, it means you have noticed something regular, something that always seems to be happening, and this means that any pattern you see is a generalization of yours. Mathematics deals almost entirely in generalizations.

Which is what numbers are: as generalizations, they are abstract objects. Very early on in mathematics, young children are asked to do things with lots of abstract objects, like those we call '3' and '10'. Not '3 pens', or '10 sweets', or '3 friends'. Just '3' or, even more curiously, the two-digit '10' by itself. Notice how quickly children are asked to add and subtract these abstract objects to and from each other. There are high expectations of every child from the very beginning in mathematics.

The central problem: communicating with, and about, abstract objects

How is it possible to communicate about abstract objects? And, since thinking is communicating with ourselves, how is it possible to *think* about abstract objects?

Notice that 'abstract' does not mean 'imaginary'. We can easily picture imaginary things, such as unicorns and beings from outer space, but abstract things are different. Abstract things include qualities like 'truth', 'intelligence', 'common sense', as well as generalizations such as '6 of anything'. The problem is, as soon as you try to picture '6 of anything', you find you are imagining '6 of something'.

The answer, as Jerome Bruner observed, is that we think about abstract objects with symbols; in the case of numbers, we do it with numerals.

The important thing about symbols is that they do not attempt to show literally what is being talked about; they are simply arbitrary, conventional marks on a page (or spoken words). When what we are communicating about with these symbols is abstract, that absence of a picture is inevitable; how could we possibly picture something that is abstract?

It is easy to picture '3 pens' or '10 friends', but what might the abstract '3' look like? Or, how about the curious two-digit '10'? Since numerals do not look like the abstract things they 'stand for', how are children to learn to interpret such arbitrary symbols in their thinking and communicating? Doing mathematics – thinking and communicating about abstract things with symbols – is certainly not easy for young children.

How does Numicon help?

Essentially, Numicon does two things. Firstly, Numicon acknowledges that in order to understand what numbers are, children have to generalize. Secondly, Numicon follows Bruner's advice in using children's actions and imagery to prepare for their use of mathematical symbols in their thinking and communicating. In practice, these two things are done at the same time.

In Bruner's terms, *enactive* and *iconic* representations (action and imagery) are used to inform children's interpretation of the *symbolic* representation (e.g. numerals) that is necessary for communicating their pattern spotting (generalizing). To aid children's necessary generalizing, strong focus is placed upon the use of structured materials.

Generalizing and reasoning – an early years example of teaching with Numicon

Initially, in common with most teaching, Numicon involves a wide variety of everyday objects (such as beads, cubes, pegs and counters, see Fig 1) in order to help children develop their counting, before then introducing the challenges of calculating.

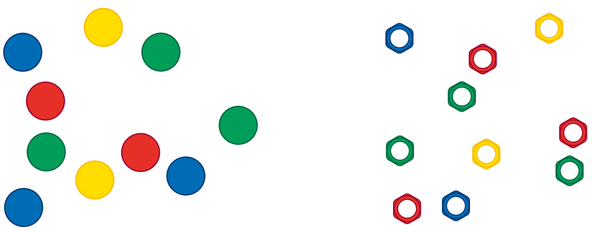


Fig 1

Importantly, Numicon also introduces sets of structured materials in which individual pieces have *regular* physical relationships with each other, for example, Numicon Shapes and number rods (see Fig 2). Children explore the physical relationships within these structured materials by, for example, ordering pieces, comparing them, combining them physically to make others.

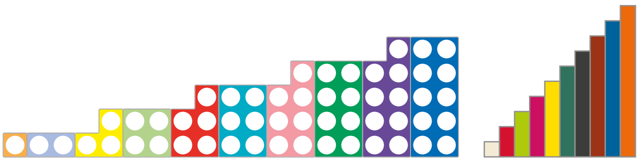


Fig 2

Thus, as children work with loose collections (beads, cubes) and with structured materials (Numicon Shapes and number rods), they are able to combine being active with physical objects and images as illustrations in their talking and thinking about numbers of things.

Numerals are introduced in association with loose collections of objects and with structured materials as children work; thus, symbolic representation appears alongside children's ongoing action and imagery with physical objects.

Number lines (e.g. Fig 3) are introduced to further illustrate the ordering that is evident within the structured materials, and to reinforce the associated ordering of numerals.

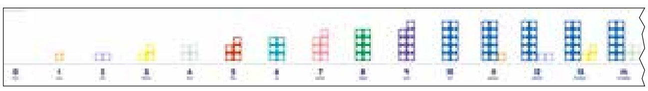


Fig 3

Importantly, loose collections of objects are arranged into the regular patterns of the Numicon Shapes; individual number rods are found to be equivalent in length to different multiples of the smallest 'unit' rod (e.g. Fig 4).

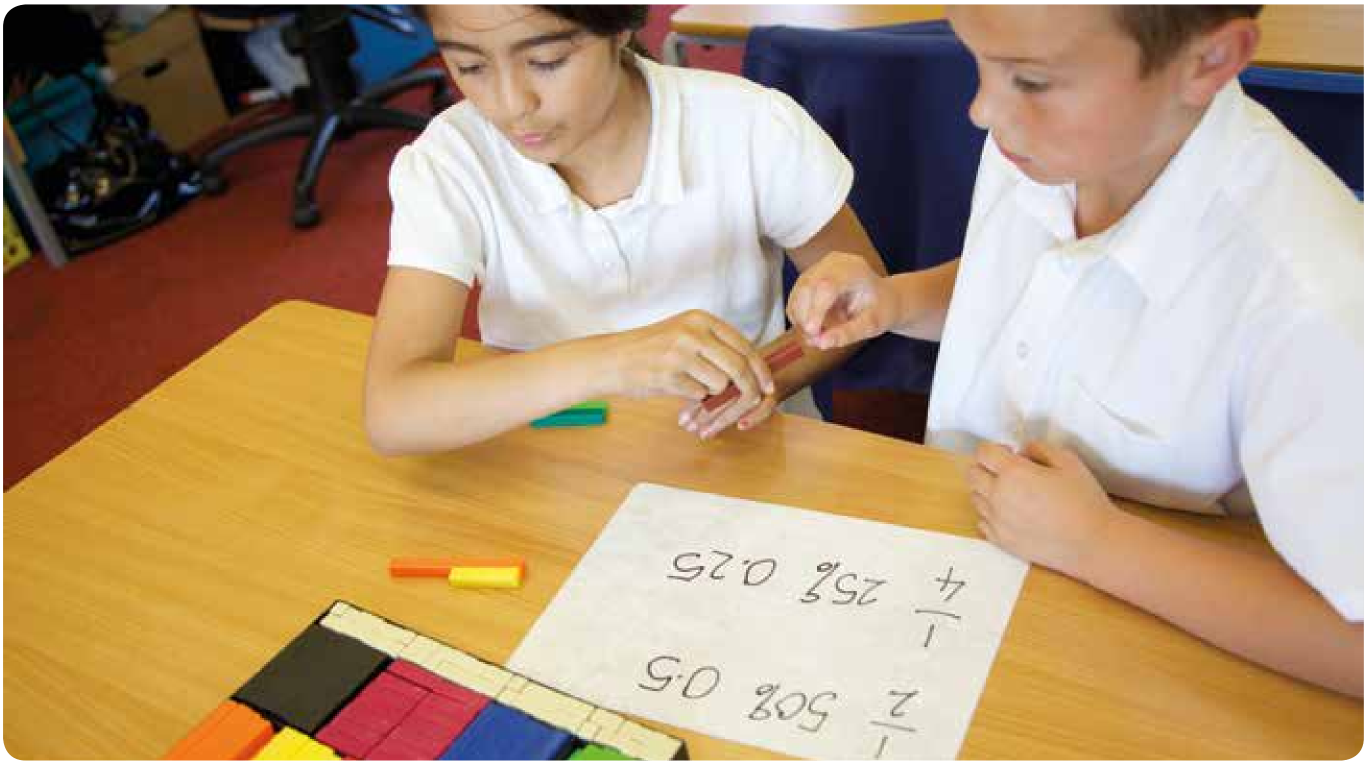
Fig 4

Through doing these activities, children learn that *any* collection of loose objects can be arranged into Numicon Shape patterns that can then be 'read'. Children learn to 'see' how many objects are in a collection without counting; numbers of things begin to 'take shape' visually, in organized ways.

It is now possible to prepare children for further generalizing about numbers through exploring relationships between numbers of things. Children's mathematical thinking and communicating continue to develop through their being active with the objects they are given, and in illustrating their talking about what they see and do.

Children generalize that any collection of loose objects can be arranged into Numicon Shape patterns. They also generalize that any number of 'unit' number rods may be exchanged for (will be 'as long as') one or more of the larger number rods. Thus *any* number of loose objects can be converted into (is equivalent to) one or more Numicon Shapes or number rods.

In realizing that any collection of loose objects can be arranged into Numicon Shape patterns, and that any number of unit



cubes is equivalent to one or more number rods, children become able to 'see the general' in any particular illustration; Numicon Shapes and number rods can be used to illustrate, in an organized way, any numbers of any kinds of things.

Numicon Shapes and number rods themselves may now be used to explore and to communicate about number relationships in general. They have become communication mediators in discussions about numbers and their relationships.

As a particular example, the Numicon Shape that has three holes fits together physically with the Shape that has five holes. The result 'forms the same shape as' (is equivalent to) the Shape that has eight holes, see Fig 5.

Similarly, the number rod worth three units, combined end-to-end with the rod worth five units, are together as long as the rod worth eight units, see Fig 6.

When laid end-to-end along a number line or number track, the '3-rod' and the '5-rod' together reach the position marked '8' on the line.

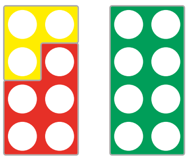


Fig 5

Fig 6

From these actions, and with these illustrations, children are able to generalize that: three *anythings* and five *anythings* together will *always* make eight things.

This generalization can be expressed using the numeric and verbal symbols: 3 and 5 together are equivalent to 8.

Later on, using further actions and further illustrations, children become able to interpret and use the further symbols '+' and '=' to express their generalization thus:

$$3 + 5 = 8$$

Importantly, at this stage children will have begun to use number words (one, two, three) as *nouns* instead of as adjectives (two sweets, three pencils) in their talking.

With their use of number words as nouns, numbers as *abstract objects* have now appeared in children's mathematical thinking and communicating, referred to with *symbols*.

Such generalizing and use of symbols can now be exploited further. If 'three of *anything*' and 'five of *anything*' together always make 'eight *things*', then:

3 tens + 5 tens	=	8 tens
3 hundreds + 5 hundreds	=	8 hundreds
3 millions + 5 millions	=	8 millions

or

30 + 50	=	80
300 + 500	=	800
3 000 000 + 5 000 000	=	8 000 000

Such is the power of generalizations, and of the symbolic notation that children can by this stage use to think and communicate mathematically about them.

Progressing from such early beginnings

The foregoing example also illustrates how Numicon supports the teaching of children's subsequent mathematics.

In a wide variety of contexts, children are offered opportunities to be active and to illustrate their talking about any (and all) relationships they are exploring between both physical and abstract objects. They do this in order that they may generalize and, as a consequence, think and communicate about their generalizations with the conventional symbols of mathematics: in other words, that they may *do* mathematics.

Of course, there is much more beyond this first example. Children have many other relationships to explore, further generalizing to do, and thus many more conventional symbols to learn to interpret and use. However, wherever it is used, the approach is essentially the same. *Enactive* and *iconic* representation are used to inform children's interpretation of the *symbolic* representation that is necessary for thinking and communicating about their constantly developing pattern spotting (generalizing).

The going gets harder as children progress in their number work: place value, ratios, fractions, decimals and negative numbers will all challenge children's capacity to generalize onwards from these early beginnings. However, such advanced work is equally possible with the same approach of actions and illustrations building towards generalizing, and a consequent use of symbols to think and communicate about the generalized, abstract mathematical objects created. This is how the symbolism of mathematics becomes meaningful.

In their work on geometry, the generalizations that children reach in their mathematical thinking and communicating happen more gradually as they progress towards being able to reason about invented abstract objects such as 'any triangle' and (later) 'any polygon'.

It is impossible to draw the abstract object 'any triangle' in the same way that it is impossible to imagine 'six of anything'. As soon as you draw a triangle, however you have chosen to draw it, you have drawn a particular one; it does not matter whether you draw one that is right-angled, isosceles, equilateral or scalene, what you have drawn is not a general triangle – it is a particular one, e.g. Fig 7.

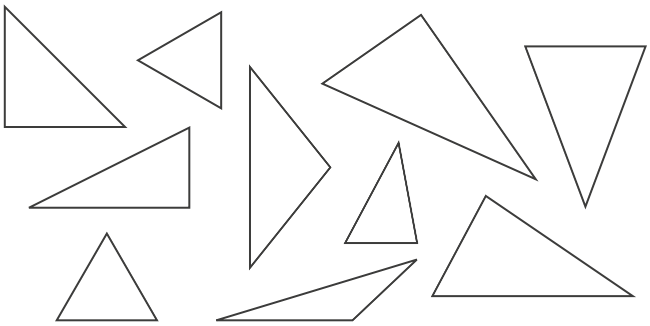


Fig 7

However, as with generalizing about numbers, in doing geometry, much exploring of relationships with action and imagery (enactive and iconic representation) prepares children for reasoning meaningfully about 'any triangle' with symbolic representation (words and symbols).

In this example, as children physically construct and transform many varieties of triangles with dynamic materials, what they see before their eyes is how lengths of sides may vary endlessly as angles change, how angles may vary endlessly as lengths change and yet, despite all the variety they see, the shapes they are creating all turn out to be triangles.

Dynamic mental imagery deriving from these physical experiences then allows children to speak and to think of 'any triangle' as they imagine an infinitely flexible closed 2D shape with three straight sides.

By stressing the straightness of the sides, that there are always exactly three sides and that the sides fit together to 'close' the shapes they make, and by ignoring the constantly changing lengths of sides and sizes of angles, children become able to imagine 'any triangle' and to reason about it with words and symbols.

Then, having generalized to a stage when their mental imagery allows them to imagine and speak meaningfully about 'any triangle', children are in a position to further generalize that, for example, 'the angles of any triangle add up to 180°' through further action, imagery and reasoning with symbols.

In the same way that generalizing about numbers is approached, exploring relationships with action and imagery in geometry prepares children for reasoning meaningfully about abstract mathematical objects with symbols.

We reach all the generalizations of mathematics, and the symbols we use to express them, through being active and illustrating our talking about the relationships we are exploring.

Doing mathematics in the world – solving problems

Of course, *doing* mathematics in the everyday world is not simply about making generalizations and using symbols. Crucially, it also involves making *use* of generalizations to solve problems in particular situations.

For example, the generalization '4 × 25 = 100' allows us to predict that the perimeter of a square of side 25 cm will be 100 cm, that the area of a field measuring 4 m by 25 m will be 100 m^2, and that if you save £25 a week for 4 weeks you will have £100. It can also be very useful to help calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

Children need to be able to readily connect the generalizing that they do with mathematical symbols with everyday world situations in which those generalizations are useful.

In their more general problem solving, children learning, for example, *how* to divide one number by another, need to learn *when* that operation is useful.

The Numicon teaching materials organize activities into groups based around mathematical themes. For example, in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, 'Estimating and rounding' (Numbers and the Number System 4) and 'Written methods of multiplying' (Calculating 12). Each activity group is introduced with, or involves, a context in which that mathematics is useful.

The activities in 'Estimating and rounding' involve discussion of everyday contexts in which we use 'round numbers' in contrast to other situations in which we try to be very accurate.

'Written methods of multiplying' begins with the use of arrays of squares of turf needed for a garden and how they help to accurately find quantities.

In the *Geometry, Measurement and Statistics 5 Teaching Resource Handbook*, Measurement 1 children are encouraged to convert between metric and imperial units in the context of building a new playground.

In these ways, children's mathematics is introduced, wherever possible, within contexts in which that mathematics is useful. Such contexts help children to 'see the point' of the mathematics they are learning, and prevent mathematics becoming a series of answers to problems they have never had.

Flexibility, fluency and persistence

Being able to do mathematics effectively also includes being able to remember basic things such as those generalizations known as 'times tables' and 'number facts'. There are many practice activities built into the Numicon written materials that encourage children to develop a regular familiarity with basic facts.

Even more important to effective functioning is flexibility in mathematical thinking and communicating. Three kinds of flexibility are especially useful.

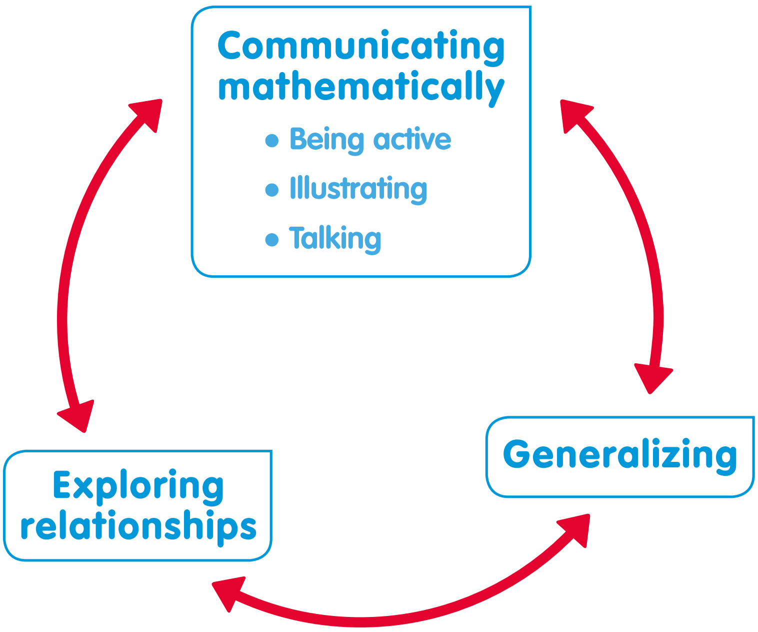
Through being active, children are able to 'invert' their actions; put more simply, this is about 'doing and undoing'. As children have found 'how many' objects there are in a collection by physically grouping them into tens (and hundreds), they are later readily able to 'partition' numbers ('undo' their groupings) when calculating with symbols. As children can combine and separate Numicon Shapes and number rods physically ('do' and 'undo' their actions), they are able to connect adding and subtracting as inverse operations and to check a subtracting calculation by adding.

The introducing and encouraging of a variety of ways of calculating means that children are able to choose methods of calculating that suit the particular numbers involved, rather than adopting a standard method for any calculation. Who would want to subtract 1998 from 4673 using a column method, just because the numbers are big?

As Numicon approaches children's use of mathematical symbols with prior actions and imagery, if children become 'stuck' or hesitant while working with symbols, it is always possible for them to recall and return to the supporting activity and illustration from which their original generalizing arose. For example, if they are working on prime numbers, using symbols, they can immediately return to actively illustrating how factors 'go into' their various multiples physically with Numicon Shapes and with number rods. They can follow a similar pattern with any other kind of number relationship. This flexibility is one of movement backwards and forwards between Bruner's *enactive*, *iconic* and *symbolic* forms of representation in their thinking and communicating.

Finally, because Numicon continuously emphasizes communicating mathematically, it is possible to stress to children that all is not lost when they feel 'stuck'. The thing to do, always, when they 'don't know what to do' in a situation is to communicate.

Being active, illustrating and talking about the relationships we are exploring *is* doing mathematics. Persistence – an invaluable quality when doing mathematics – comes from continuing to communicate, with yourself and/or with others, whenever (for the moment) you cannot see where you are going.



Preparing to teach with Numicon Number, Pattern and Calculating 5

This section is designed to support you with practical suggestions in response to questions about how to get started with Numicon in your daily mathematics teaching. It also contains useful suggestions on how to plan using the long- and medium-term planning charts as well as information on how to assess children's progress using the Numicon materials.

When beginning to work on Numicon Number, Pattern and Calculating 5, as well as reading this section, it can be helpful to refer to the suggested teaching progression in the programme of Numicon activities. This, along with the long- and medium-term planning charts, can be found in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*.

In this section you will find overviews of:

How might using Numicon affect my mathematics teaching?

There will be a constant emphasis on encouraging children's mathematical communicating, especially when the going is tough. This is because the mathematical communicating is the 'doing' of mathematics. The increased focus on mathematical communicating will make it easier to judge whether children are facing a suitable level of challenge through watching, questioning and listening to them as they work through activities. Numicon aims to encourage children to relish challenge, have a sense of achievement when a challenge is overcome and be excited about and ready for the next one.

As the focus on mathematical communicating grows, you may become more aware of the words and terms you use in your teaching. It is important to use words and terms consistently. Children will be meeting new mathematical language in Number Pattern and Calculating 5 as they meet new and often challenging ideas. They will need time to assimilate these unfamiliar words and terms before they are able to use them fluently in their own communicating. Try to encourage other adults in your class – and throughout the school – to illustrate the consistent use of mathematical language.

Consider the imagery being used in the classroom. Listen out for children using these same words and imagery to explain their ideas, though to start with they may use them only hesitantly. As communicating becomes established as part of the culture of the classroom, children will increasingly join in conversations with you and their classmates and mathematics lessons will develop into dialogue with ready use of structured imagery to illustrate ideas.

A sense of shared endeavour will develop in the class as children persist through difficulty to solve problems. So, when children know they are 'stuck', they will know that the thing to do is talk about it, to try to explain what the difficulty seems to be and to use illustrations and actions to express the problems. Careful questioning can encourage children to really think about 'difficulty' and to naturally come to solutions.

How can I encourage communicating?

Children respond to the examples around them. As such, the ways in which you communicate mathematically provide a model for children's communicating. Engaging in dialogue with children – actively listening to what they are saying and responding sensitively with thoughtful questions – will encourage children to listen to one another and respect each other's ideas. Encourage children to read, spell and pronounce mathematical vocabulary correctly.

Make sure that materials and imagery (e.g. Numicon Shapes, number rods, number lines, base-ten apparatus) and other communication mediators (e.g. counters, interlocking cubes) are freely accessible. Observe how children use these, listen to what they are saying, watch what they are doing and

respond with questions and qualified praise when you notice active listening and thoughtful questioning. What children do and what they say will help you to understand what they are thinking. Children will also use jottings, drawings (e.g. empty number lines) and formal written methods to communicate their ideas – and will become more adept at organizing their notes systematically as they discover that well-organized notes help them to explain their ideas clearly.

The ways in which children are grouped, or paired, for working together has an impact on their communicating, so this also needs careful consideration (see page 28, 'What about grouping children?').

Using daily routines to encourage communicating

Introducing a 'morning maths meeting' has been proven to be very successful in encouraging children's mathematical communicating. These can last for about fifteen minutes, usually at the start of each day.

These meetings are oral and practical and include a small selection of key routines every day in which children practise recall and gain fluency with the ideas and techniques that are the focus for that term. These might include: counting practice (e.g. counting on or back from a number within the broad counting range for the class, counting in different multiples, counting along a fraction, decimal or negative number line); practising recall of number facts (e.g. multiplication facts for 2 to 12 times tables, dividing facts, complements to 100); discussing observations about a mathematically rich object that you or a child has brought in; solving a mathematical problem that has come up in the class, in the school, in the news, at home or in a story.

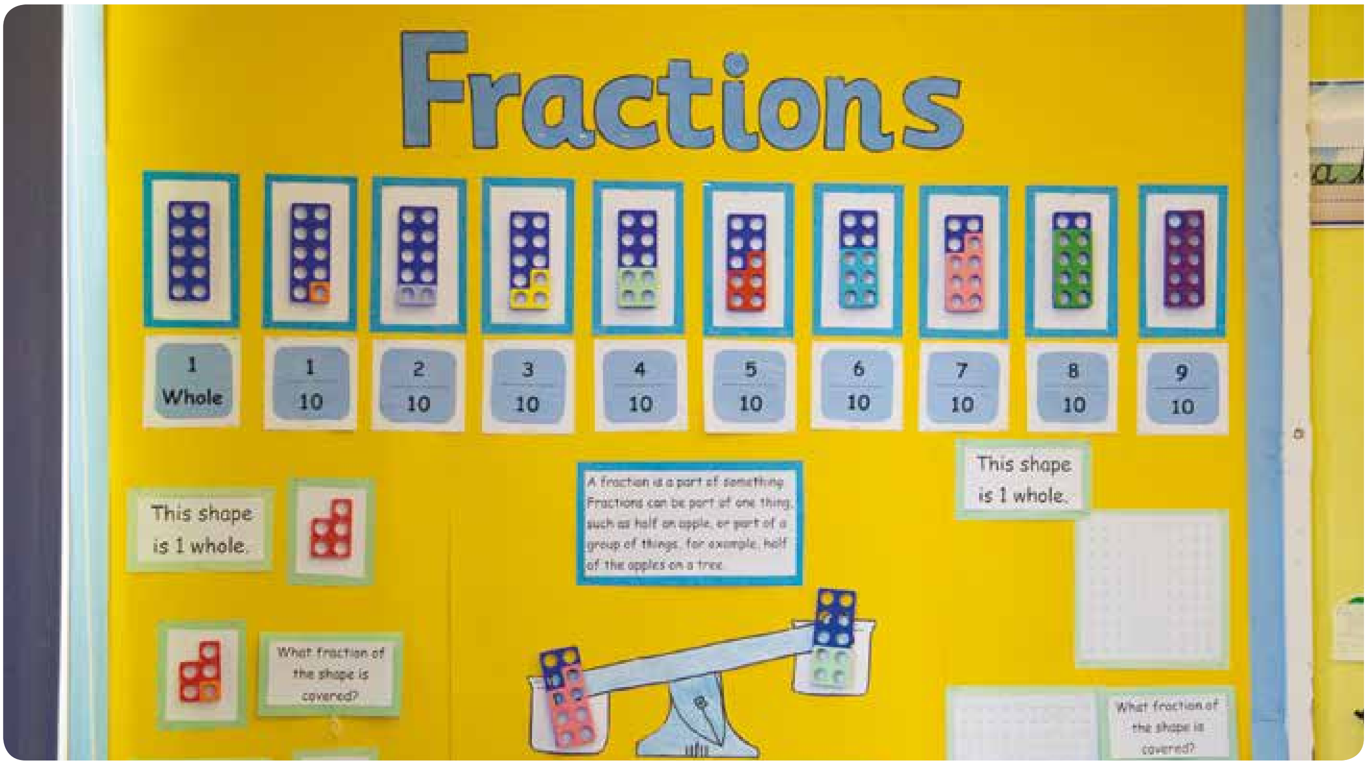
You can refer to some of the whole-class practice activities from the activity groups in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* for ideas for 'morning maths meetings', or you could also use one of the focus activities from the activity groups for problem solving. You could also create your own problem-solving questions for children to answer, e.g. 'Roughly what fraction of the class is having school dinners today?'

Children's mathematical thinking and reasoning can be encouraged at different times during the day. For instance, when children need to stand in line before going out to play or going to assembly, provide them with statements that will require them to think carefully. You could say: 'If your birthday is in the seventh month of the year, join the line'. Vary with other suitable statements that will test children's reasoning, e.g. 'Join the line if your birth date is a multiple of 7' (or any other multiple from the 2 to 12 multiplication tables), or 'Join the line if your age is a factor of 63'.

Using daily opportunities and holding 'morning maths meetings' helps to ensure that children experience the breadth of the mathematics curriculum for each year and that they do not see mathematics as something that only happens in specific mathematics lessons.

What might the use of Numicon look like in my classroom? Supporting communicating with a number-rich environment.

Numicon classrooms should be rich in number illustration and visually show that mathematics is an important part of the children's everyday learning experience. Numicon images and numerals can be incorporated into labels and displays



in many areas of the classroom, inviting children's attention. There will be number lines and charts on display where they can easily be seen, including a Numicon 1 000 000 Display Frieze, a Numicon 0–1·01 Decimal Number Line, a Numicon $^{-}$12–12 Negative Number Line, a Numicon 0–100 cm Number Line and a place value chart. In a corridor in the school, there may also be a Numicon 0–1001 Number Line to engage with.

At different times during the year, take the opportunity to put up mathematics displays that celebrate children's work including displays around a particular aspect of mathematics such as dividing, percentages, fractions, temperature or geometry. These may include number lines, pictures, books and interesting objects, as well as children's work. It is useful to set up a mathematics table with an interactive display, where children can freely explore Numicon Shapes and number rods in practice activities that they have first met in focused teaching sessions. This might be finding a mystery number by following clues, or perhaps one of the challenges from the activities in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 6 'Logic and reasoning'.

All the classroom resources will be organized systematically, with storage trays numbered and stored in a logical order. Mathematics equipment will be easily accessible to children, so they can quickly find equipment when they need it. For children who are new to Numicon, you might set out equipment on children's tables for them to use for specific activities.

This mathematically rich environment itself provides children with valuable learning opportunities. Nearly all of us are acutely visually aware, and children are no exception. You will often notice them looking thoughtfully at number lines or other imagery as they think (communicate with themselves). Throughout the day, and particularly when you are teaching the Numicon activities and solving mathematical problems, you will find children referring to the imagery and displays.

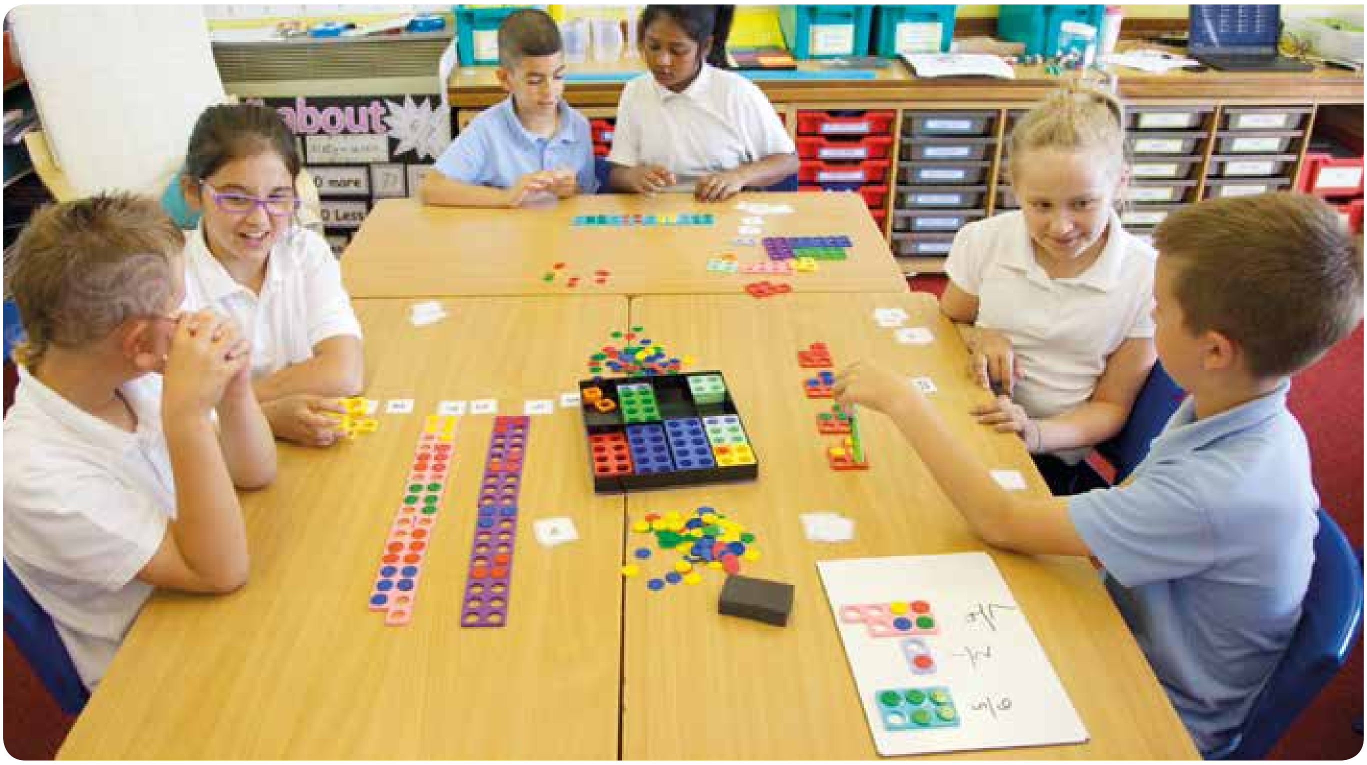
What could using Numicon feel like for children in my class?

By illustrating number ideas with materials and imagery in your teaching, such as Numicon Shapes, number rods, number lines and base-ten apparatus, numbers become real for children. They also become aware that mathematics is something useful that they do at many points throughout the day. As you engage them in solving mathematical problems, they learn when to use the mathematics they are learning. Therefore, they do not see mathematics as something they do only in their daily mathematics lessons.

Children also feel supported by the imagery on display; you are likely to notice them glancing at relevant displays to check an idea they are explaining. At other times, you may notice them looking thoughtfully at imagery – they are likely to be noticing relationships and making connections as they assimilate new ideas.

Children look forward to working with materials, and engage with Numicon activities whose practical and open-ended nature invites them to experiment. While exploring an activity, children move the apparatus around and self-correct as they seek solutions. These opportunities to self-correct support their confidence.

The emphasis on communicating in the Numicon activities means that often children are working with others, often in



pairs within larger groups. Working with others also supports children's confidence by providing ample opportunity for them to share ideas and work things out together. When they are working with a partner, many children appear confident, setting each other challenges and exploring ideas more deeply than they do when they are working alone.

Children take their lead for listening to each other and sharing ideas from the ways in which the adults in the classroom converse with them and each other. They will need help with taking turns, showing respect for each other's ideas, listening to one another without interrupting, and with ways of phrasing questions and expressing ideas. Over time, you will find that children become increasingly confident in sharing their thinking.

Children also feel confident when tackling new ideas because they can use materials and imagery to work out ways through a problem. Their confidence is further encouraged by your interest as you discuss their ideas with them, helping them to become consciously aware of what they know and are learning, as they explain their thinking. This discussion supports children's monitoring of their own learning.

Is there a risk of children becoming over-dependent on the imagery?

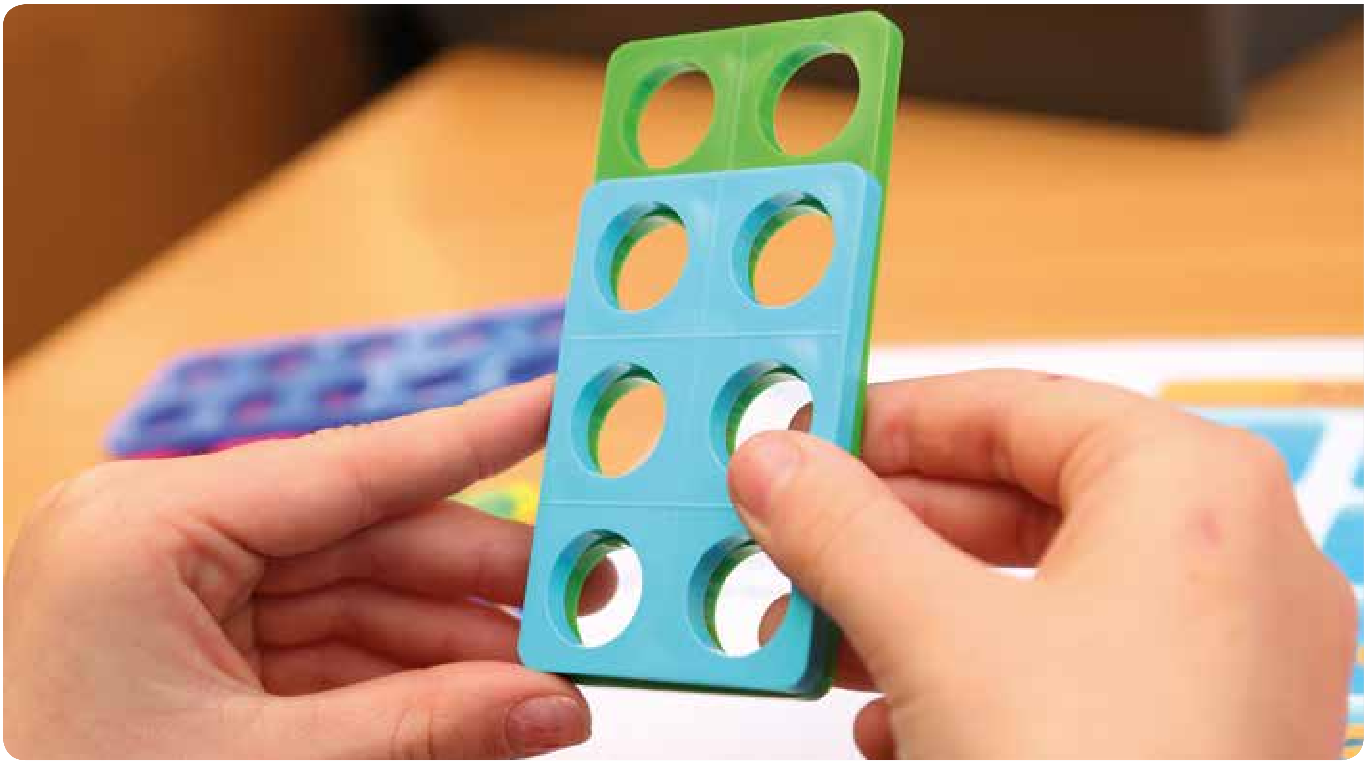
Structured apparatus, such as Numicon Shapes, number rods, base-ten apparatus and number lines, offers children structured illustrations of number relationships in ways not provided by written symbols. Children who are new to Numicon can be helped to develop mental imagery of number relationships by following activities with the Numicon Feely Bag described in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Getting Started, in which children identify Numicon Shapes by touch, and group objects into patterns of all the Numicon Shapes without counting.

You will find that children like to work efficiently and quickly and, once they are secure with an idea, they tend to move on to working in their heads, using their own mental imagery. However, when children are consolidating an idea, or meeting a new one, they will often use the physical materials and imagery until they are well on the way to understanding.

Thus, there is no point at which imagery becomes redundant; meeting new mathematical ideas is ongoing and illustrating thinking with structured imagery gives children of all ages and abilities a practical, accessible way to investigate mathematical problems.

How much time should I plan to spend on mathematics teaching?

The time spent teaching mathematics can vary as there are many opportunities to discuss mathematics across the school day. In addition to a daily mathematics lesson lasting up to an hour, and a 'morning maths meeting' lasting about fifteen minutes, there will be opportunities for developing language of comparing, position, time and shape that will inevitably crop up. There will also be many real opportunities for calculating, such as working out what percentage of the day is spent in the classroom or working out the number of exercise books and pencils needed for the class for a term or year. Try to use current events in the school as well, e.g. calculating how much each class has contributed to a school sponsored event and then calculating the total; working out how many tickets each family might be allocated for the school performance.



Solving these sorts of problems helps children to realize that doing mathematics is useful in all sorts of situations and gives them opportunities to use the mathematics they are learning as they work out what they need to use to solve the problem.

The depth and reach of the mathematics that children are meeting in Number, Pattern and Calculating 5, e.g. fractions, decimals, more algebra, and the subsequently extended nature of the activities, mean that, at times, children will need longer to work on them. For some children this may mean time to take an activity further, and for others time to practise and consolidate ideas. This time may be outside the usual mathematics lesson.

What format might Numicon mathematics lessons take?

There is no hard and fast rule about the format of your mathematics lessons, as this is likely to be flexible, changing according to what you are teaching and how the children respond. Often, the first part may be a whole-class introduction including an oral and mental 'starter', followed by a longer session of group work during which children are either working independently or being taught in a focused teaching group. Finally the class usually comes together for a concluding session. Throughout the lesson it is a good idea to allow plenty of time for discussion.

During the introductory session, you are likely to use Numicon Shapes, number rods, number lines, grids, base-ten apparatus and other imagery in your communicating with children. You may also use the *Numicon Software for the Interactive Whiteboard*. Children may join in this whole-class part of the lesson in lots of different ways: participating in a class conversation; talking with a partner or discussing in a small group; jotting on an individual whiteboard; or holding up Numicon Shapes or number rods to show their ideas.

When you are introducing a new activity group to children (often at the start of a week) you may find that the children work as a whole class throughout the lesson, with lots of discussion, and that any recording is done quickly on whiteboards.

In the second part of the lesson, children will be arranged in groups working on, for example:

- a teacher-led focused teaching activity
- a teaching assistant-led activity
- an independent practice activity or investigation, or further work on ideas introduced at the start of the lesson.

Groups of children will be exploring ideas from the same activity group but are unlikely to be working on an identical activity. The various groups may be using different apparatus, e.g. for a lesson on fractions, some groups may be using Numicon Shapes, others may be using number rods, others may be writing and drawing number lines, number sentences and diagrams.

Over the course of a week, the different groups of children will rotate around the various group-work activities, so that all children receive focus teaching and explore the ideas using different imagery.

In the final part of the lesson, it is particularly important to encourage all children to reflect on their learning by asking questions of children working at different levels, depending on what you have noticed them doing and saying during the

lesson. You may ask the different groups to explain to the rest of the class what they have been doing and what they have noticed. You may also have points you want to draw to children's attention. You could suggest what might happen in the next lesson and anything children could think about before then. To help children reflect on their lesson, you may ask them to think quietly for a few moments about what they have been doing and guide their reflection with questions, such as:

- Is there anything new you have learned?
- Is there anything you feel particularly pleased about?
- Is there anything you particularly enjoyed?
- Is there something that you found difficult?
- Is there something that you are still puzzling about?
- Is there something you would like to do again?

This is not an exhaustive list, but by encouraging children to reflect, you will help them to become consciously aware of what they know and to begin to monitor their own learning.

How will I manage the class for the mathematics lessons?

There are several important factors to consider for organizing and managing successful group work.

Firstly, to engage children, it is important to plan differentiated activities that will provide appropriate levels of challenge for all children. Therefore, during the week, the group-work activities will need to be adjusted for different groups.

Secondly, as children may be working on activities that have been appropriately adjusted to make them suitable, it is important that they are comfortable with how to do the activity and what is expected of them.

Thirdly, the order in which the activities are introduced to different groups has an impact on how quickly children progress with their activities.

At the beginning of the week, the class may be taking in a lot of information, so give consideration to how you will introduce the activities as this will impact on how quickly children can access and begin to make progress through the work. You will find ways that work well for you, but the following guidance may be of help:

- Explain the simplest independent work first.
- The first time you introduce a challenging practice activity, allocate it to a group of children who are able to follow instructions well.
- Groups that might need more support should begin the week with a focus activity; the adult working with the group can explain the activity, removing the need to spend time explaining it to the whole class.

Once the activities have been introduced, children will then go off to work in their groups. Groups working independently on practice activities will work either individually or in pairs or small groups. As the teacher, you will find that you will be able to focus on one or two groups. If you have a teaching assistant, they will focus on another. Best practice suggests that the teaching assistant works with all the different ability groups at different times, as does the teacher, helping to keep expectation of achievement high and reducing the likelihood of children becoming over-reliant on adult support.

During this part of the lesson, it is quite likely that a child or children will put forward an idea that is worth everybody considering. In which case, you might choose either to invite all children to take a moment to reflect on the idea, or make

a note to discuss this in the final session when the class comes back together.

Over the course of time spent on an activity group, different groups of children are likely to be exploring ideas from the same activity group but are unlikely to be working on an identical activity. Over the course of a week, the different groups of children will rotate around the various group-work activities, so that all children receive focus teaching, and explore the ideas using different imagery.

The depth and reach of the mathematics that children are meeting in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, and the subsequent extended nature of the activities mean that, at times, children will need longer to work on them. This time may be outside the usual mathematics lesson, or you may choose to spread one area of work over several lessons.

What writing or drawing might children do in mathematics if the activities are mainly practical?

Writing and drawing are aspects of children's communicating and will take many forms. Bear in mind that this is their communicating, so it may be idiosyncratic. Children are now using written methods for column adding and subtracting and for multiplying and dividing, which become an important part of their communicating.

There are opportunities signalled in the activities for children to communicate on paper wherever this serves a useful purpose. Amongst the photocopy masters there are some activity sheets, but in general it is recommended that children write and draw in their mathematics exercise books. This provides a useful bank of evidence that shows how children's communicating is developing over time.

Children's Explorer Progress Books (see page 9 for more information) will also provide an extremely useful source of evidence for monitoring how children are progressing throughout the year.

Giving children the choice of how to communicate their ideas can provide useful insights into how they are approaching problems, how comfortable they are with formal written methods, whether they are working systematically and how they are using conventional notation.

What about organizing resources?

It is worth considering that the way you organize and use the resources yourself gives a strong message about how you would like children to use and care for them.

As you plan and prepare your mathematics lessons, check the necessary equipment by referring to the '**have ready**' section in the activities you plan to teach. The photocopy masters can be photocopied from the *Number, Pattern and Calculating 5 Teaching Resource Handbook* or printed from the Numicon Planning and Assessment Support.

Decide whether to set the equipment on children's working tables or to ask the children to collect what they need for themselves. Children who are used to working with Numicon apparatus can collect the essential equipment they will need for an activity, or you may decide to provide them with a list, leaving them to think for themselves about any further equipment they may decide to use. Alternatively, to save time, and for children who are new to Numicon, you may

sometimes decide to set the equipment on children's working tables. If you have planned for groups of children to work on each activity over a week, you can save time by sorting the equipment needed for each activity into a tray or basket and storing ready for use each day.

What about grouping children?

By the time children are working on activities from the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, some children will be comfortably fluent with the mathematics they have learned in previous years; they will be able to engage confidently with the new ideas they are meeting and communicate their ideas clearly. Others will need more time to consolidate their understanding of ideas met earlier, plenty of practice to develop fluent recall of key number facts and more exposure to the new mathematical words and terms they are meeting, before they are able to communicate their ideas confidently. These differences need to be taken into account when grouping children and planning lessons.

Some schools respond to this by putting children into sets of similar ability in an attempt to create homogenous teaching groups and eliminate the need to differentiate tasks. In fact, there will still be a range of needs even within sets of similar ability and therefore the need to differentiate remains.

It is important to be aware that there are risks involved in ability setting that can have negative impacts, such as: of putting an artificial ceiling on expectations of children's ability and achievement; of creating an air of complacency in higher-ability sets; of children in lower-ability sets seeing themselves as failing in mathematics. Children placed in either a high- or low-ability group, irrespective of their ability, are likely to take on characteristics of that group: misplacement can therefore result in able children underachieving.

Many schools have moved away from setting, finding that mixed-ability classes provide opportunities to be more flexible, so children can sometimes work in mixed-ability groups and at other times with children working at a similar level; these schools also often find that this is resulting in raised achievement for all children.

As you teach the class you will discover which children work well together and their levels of understanding, and use this to adjust groups during the year, so that children have the opportunity to work with different peers.

Whatever the policy is in your school, bear in mind that it is important to vary groups from time to time to ensure that children do not always work with the same partner.

How do I prepare for teaching mathematics lessons using Numicon?

Understanding the mathematics yourself

No one carries all the details involved in doing, e.g. dividing, multiplying or place value, in their heads all the time; we all need to remind ourselves what particular areas of mathematics are about before we introduce them again to children. Before you teach an activity group, read the relevant sections from the 'Key mathematical ideas' chapter (pages 44–67) before doing anything else to prepare for your teaching. Ask your own questions. If there are things

you want to do further research on, consult other sources, for example Derek Haylock's *Mathematics Explained for Primary Teachers* (4th ed, 2010).

Mathematical communicating essentially involves generalizing, so always look for, and be clear about, the *generalizing* to be done in any activity group. As an early example, children's first introduction to numbers could helpfully encourage them to notice that:

3 beans + 5 beans = 8 beans

3 sweets and 5 sweets = 8 sweets

3 pencils and 5 pencils = 8 pencils

3 flowers and 5 flowers = 8 flowers

… and so on.

Does it matter what these things are? No. So, *generalizing* from the structure of these illustrations, we can write down:

3 of anything + *5 of anything* = *8 of those things*

Or, 3 + 5 = 8

This approach can then lead helpfully on to further generalizing …

3 tens + 5 tens = 8 tens

3 hundreds + 5 hundreds = 8 hundreds

3 trillions + 5 trillions = 8 trillions

Or, 30 + 50 = 80

300 + 500 = 800

The children may be new to any area, so another way to work on generalizations is to ask 'What *patterns* in the work they're doing will children have to notice in order to progress?'

When children notice things, be prepared to keep asking: 'Will that always work?', 'What if those numbers were different?', 'Would that work with fractions?', 'Will that ever work?', '*When* does that work?' and 'What never works?' Now you know what the mathematics in an activity group is about, ask yourself, 'When might that be *useful*?'

Appreciating the contexts

The 'Educational context' on the introductory page for each activity group will help you to see how the ideas are developed and how they fit into the continuum of children's learning about Number, Pattern and Calculating.

Think about the kinds of context offered in the activity group you're working on and ask yourself: 'Is this mathematics useful in particular kinds of 'real-world' situations, or will it help me to do some other mathematics?' It can be helpful to think up one or two contexts of your own, just to be sure that you understand what the *point* of doing this mathematics is. Children do not just need to know *how* to do this mathematics; they need to know *when* to do it, as well. How can you help them to spot when this general mathematics applies to a particular situation?

Understanding the illustrating and communicating involved

Whatever mathematics the children are doing, if they've got as far as using numbers, they are working with generalizations *they cannot see*. Consider what illustrations are available to you and the children to support work on a particular activity group. In what ways might illustrations help children to *see the general in the particular*? For instance, if you want children to generalize that 'it doesn't matter which

way round you add two numbers, the total will always be the same', then using visual Numicon Shapes or number rods as illustrations is very immediate.

Study the illustrations offered in an activity group. Consider if there are any further illustrations that could be used. Give thought to which illustrations will help children to explore the relationships best.

Think about which symbols and words are key to successful communicating in this area and whether any of these are new. Activity groups suggest key terms; the idea is that, through doing the activities, the children will learn how such terms are used. Encourage children to use them in their conversations and let them notice how you use them.

Review all the activities from the activity group you are working on in terms of the mathematical communicating involved. What *actions*, *imagery* and *conversations* are children going to be using and having during this work?

Selecting and adapting activities

Read all of the activities in the activity group and identify what each activity contributes to the overall work. Then try the activities.

You know the children of your class, and the materials available to you. You will be best placed to select which activities are most appropriate and adapt them creatively to suit the needs of the individual children.

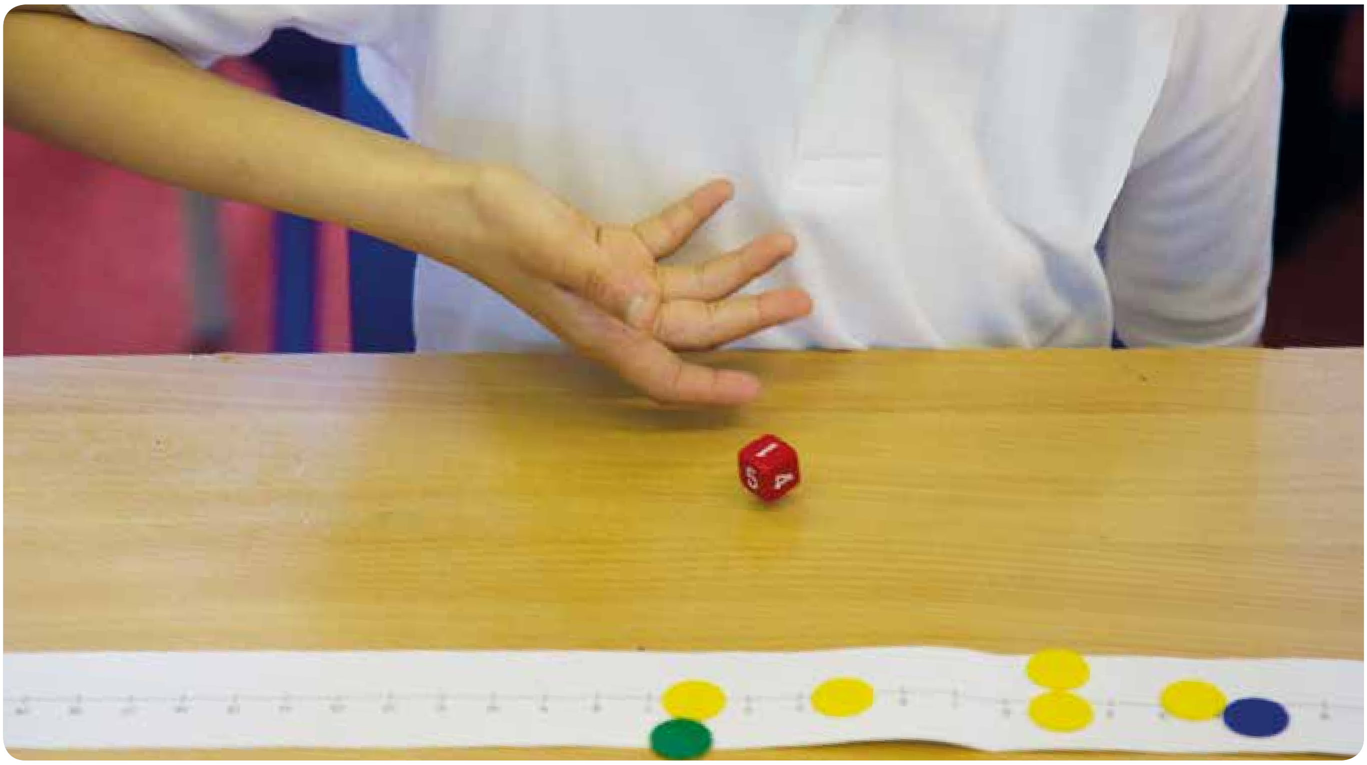
Some activities might be revision for your children; others might involve ideas, illustrations or techniques that are completely new. Some activities will be invaluable practice. You might think an activity will be too easy/difficult for some children, so consider how you might make it more/less challenging. Be flexible; adapt what is available for your children in light of what they can already do.

How do I deal with children who are stuck?

It is important for children to know that there is nothing wrong with challenge; it is quite normal to get 'stuck'. This can only be achieved when they know what to do when they are stuck.

What is important is that children are encouraged to communicate mathematically in the face of challenge and, provided the activities are suitably differentiated (so the challenge is not impossibly difficult), you can help them to do this by asking questions such as: 'Do you remember how you started?', 'Were there any parts when you just felt completely stumped?', 'Does this problem connect with any other problems you can remember?' and 'Would it feel different if we changed any of the numbers?' This will help them to explain what the difficulty seems to be, and to use illustrations and actions to express their problems. In time, children will start to feel that they can ask themselves these sorts of questions so that they become confident to persist through difficulty.

Eventually the class culture will become one in which children are confident to deal with challenge because they feel a sense of achievement when they do so. Encouraging children to express and deal with challenges will also help them to respond positively if they are 'stuck' in a test or exam, when they will need to explain the problem to themselves silently and visualize mental imagery.



Children's confidence can be supported further by following the recommended progression of teaching activities in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*. This progression is designed to build children's understanding cumulatively to try to ensure they have the understanding needed to tackle each new activity group. The ideas in the activities are introduced carefully so that the challenges children meet are just within their reach.

How do I plan in the long- and medium-term using the Number, Pattern and Calculating 5 teaching programme?

The plan–teach–review cycle applies to Numicon, just as it applies to all effective mathematics teaching. There are, however, four important features of Numicon that support this cycle.

Firstly, the Numicon teaching programme (the suggested order of teaching the activity groups) is structured progressively. This chart can be found in the long- and medium-term planning section of the *Number, Pattern and Calculating 5 Teaching Resource Handbook* (this is also available as an editable version in the Numicon Planning and Assessment Support).

The second feature is that there are practice and discussion activities within each activity group, some for individual work and others for paired work.

Thirdly, assessment can be more accurate through children's practical work with Numicon materials and imagery and communicating their ideas through talk and on paper. These assessments will, in turn, help with planning.

Finally, 'using and applying' does not need to be planned separately. This is partly because all the activity groups involve problems that need to be solved, but also because the cumulative nature of the teaching programme means that children are using their earlier ideas every time they face a new one.

The teaching programmes throughout Number, Pattern and Calculating are arranged into three strands: Pattern and Algebra, Numbers and the Number System and Calculating. Within each strand is a sequence of activity groups, though the strands are interrelated and what children learn in one strand supports their learning in another.

The long-term plan in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* shows the recommended order for teaching the activity groups. This plan has been carefully designed to scaffold children's understanding so that they are able to meet the challenges of each new idea, e.g. children would not be expected to learn how to work on decimals until they understand place value with whole numbers.

The medium-term planning guide in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* gives the expected coverage over the course of the year and also lists the activities and the learning opportunities for each group. The medium-term plan also lists the milestones that children need to be secure in as they progress through the teaching programme. These milestone statements should be used to assess children's progress throughout the year.

You may decide to follow the long- and medium-term plans as they stand. You may also find that you need to split some of the larger activity groups and return to them later.

There are summary charts showing the title and learning opportunities for each activity group in the long- and medium-term planning section of the *Number, Pattern and Calculating 5 Teaching Resource Handbook* and in the Numicon Planning and Assessment Support.

The 'Using the activity groups' section on pages 36–37 (also included in the Teaching Resource Handbook) highlights the key parts of each activity group.

Each of the activity groups begins with a 'low-threshold' focus activity, designed deliberately to support confidence and ensure that all children are included. The remaining focus activities are designed to help children progressively develop their ideas around the theme of the activity group. You will notice that there are opportunities for reasoning about numbers throughout focus activities through some quite challenging questions.

The focus activities are designed for whole-class or group teaching. Some may be taught quite quickly to the whole class as an introduction to be explored later with a focus group.

Ensure that activities are differentiated where necessary so that all children who should be working independently can do so. Include activities that allow children to become more confident and allow them to be able to work more speedily by practising and celebrating what they are able to do. As you decide which activity to allocate to different groups of children, remember to check that there is scope for children to take the activity further. You can increase the challenge through your questions with specific groups and by asking challenging questions when the class comes together for the final part of the lesson. Your questions will depend on what you have noticed the children doing and saying during the lesson.

As you plan for your 'morning maths meeting', build in whole-class practice activities from earlier activity groups to help children to develop fluent recall. There are some activity groups, for example, those where practice is needed to develop fluency with formal written methods of calculating, which children will perhaps need to revisit often over a longer period, alongside work on other activity groups either from the *Number, Pattern and Calculating 5 Teaching Resource Handbook* or from the *Geometry, Measurement and Statistics 5 Teaching Resource Handbook*. The educational context of these activity groups will highlight if extra time is likely to be needed for new and difficult ideas.

How long should I allow for teaching each activity group?

The Numicon teaching activities for Number, Pattern and Calculating 5 primarily address the number sections of the mathematics curriculum. There are thirty activity groups within the *Number, Pattern and Calculating 5 Teaching Resource Handbook*. The first activity group, Getting Started, is designed primarily for children who are new to Numicon activities. Allow such children time to cover this activity group, as it is essential to their success with Numicon that they are able to make connections between Numicon Shapes and patterns, number rods, base-ten apparatus, number names, and numerals, and are able to start using the apparatus to illustrate adding, subtracting, multiplying and dividing. They also learn the actions for the symbols for these operations (+, −, ×, ÷), and the equals symbol (=).

Each of the remaining twenty-nine activity groups can form the basis of approximately one week's work around an area of mathematics, although this will vary depending on both the activity group and the children. The depth and reach of the mathematics that children are meeting in Number, Pattern and Calculating 5, and the subsequent extended nature of the activities, mean that, at times, children will need longer to work on them. This time may be outside the usual mathematics lesson or it may take several lessons. You will also gauge from children's responses whether they need a longer timescale for some of the more difficult ideas, e.g. fractions, so it is important that you are flexible in the amount of time you allow for different activity groups.

Look through the long- and medium-term plan for Number, Pattern and Calculating 5 to assess how much time you think will need to be given to each activity group. Careful consideration will need to be given to how much of each activity group is essential for the children in your class to complete before moving on. It will also be crucial for you to leave sufficient time for the activity groups at the end of the Numicon teaching programme for Number, Pattern and Calculating 5 so that children don't miss out on content that happens later in the school year.

It is highly improbable that you would expect all the children to do all the activities in every activity group, but the detailed progression and range of focus activities is there to provide flexibility for teachers to exercise their professional judgement as to which children need to work carefully through the earlier activities in a group and which children are moving on quickly, to the later, more challenging activities. The detailed progression provides a bank of material to draw on if a child is having difficulty getting to grips with an aspect of the mathematics you are teaching.

As you teach the activities, you will find that some children will move on very quickly and you will be able to combine two or sometimes even three activities within one focus teaching session. These children may well complete all the activities in the group.

Other children may need a little longer to establish secure understanding, but will cover several of the activities and benefit from returning to finish the activity group after a week or so. This has the advantage of reminding children about ideas they have met earlier and gives you useful opportunities to review what they have remembered. For more assistance, refer to the medium-term planning guide in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*.

You will see that the progression of activity groups in the medium-term plan is punctuated at intervals by assessment milestones that occur at broadly half-term intervals. When considering which activities to utilize from an activity group, refer to the relevant milestone statements from the medium-term planning to guide you. The milestones pick out those aspects of mathematics that are absolutely essential for children's progress, and so specify what they cannot afford to miss.

What about differentiating activities?

By the time children are working on Number, Pattern and Calculating 5 there will be a wide range of mathematical achievement and aptitude within any class. Some children will already be fluent with ideas they have met in previous years, have fluent recall of number facts and be able to use these and to think mathematically when solving problems. Others will be working less confidently and be less fluent. This will have a significant impact on how you plan activities that offer appropriate levels of challenge for the varying needs of the children so they maintain their interest and reach their potential.

First of all, check the 'learning opportunities' and the 'educational context' on the introductory page of the activity group, which will give you an overview of the ideas children will be meeting, and the previous learning that it builds on. If your earlier assessments tell you that some children are not yet ready for the level of challenge in the activity group, look back through earlier activity groups in the same strand to find appropriate activities. For some children it may be appropriate to differentiate by adjusting the number range within the activities.

Each activity group starts with a 'low-threshold' activity which is designed to be accessible to all children (although in a mixed-age class you will need to modify the work for the younger children and assess how they respond). You may decide that other children are ready to go straight to the more challenging activities later in the activity group, and you will find the open-ended nature of the activities and the emphasis on mathematical thinking means that there is always room for children to take activities further.

For the highest-achieving children, you may decide to increase the challenge further through planning specific questions that extend the reach of the activity.

Those children who take longer to process and assimilate new ideas may need to work for much longer on the earlier activities in the group (or even activities from an earlier activity group in the *Number, Pattern and Calculating 5, 4* or *3 Teaching Resource Handbooks*). For these children, you will return to the later activities in the activity group when the class is working on another activity group on a similar topic, e.g. if children are not confident with the strategies for bridging when adding and subtracting in *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Calculating 2, you may decide to do more work on this when the majority of children are doing further work on adding and subtracting in Calculating 3.

How can I support children to develop fluency?

At the end of each activity group is a list of suggestions for whole-class and independent practice and discussion activities to help children to develop fluent understanding of the ideas they are meeting in the activity group. You can select from these to give children appropriately differentiated opportunities that will help them to develop fluency and confidence with the mathematics they are learning.

How successfully children deal with the ideas they are meeting in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* depends largely on how fluent they are with ideas and number facts they have met previously, particularly multiplying and dividing. If children are not yet

fluent with earlier ideas, then help them by selecting from the suggestions for whole-class and independent practice and discussion from earlier activity groups (if necessary referring back to the *Number, Pattern and Calculating Teaching Resource Handbooks 4* and *3*). A gauge of children's fluency is whether they are able to use their knowledge and understanding flexibly to solve new problems that may be set in a different context from those used in earlier activity groups.

In the weeks after you have taught an activity group, continue to encourage children's fluency by drawing on these suggestions for whole-class practice and continue to set problems. This can be done during 'morning maths meetings', at odd times of the day, or in the mathematics lesson.

Explore More Copymasters provide further opportunities for children to practise and discuss at home the ideas they have been working on at school.

For further guidance on how Numicon addresses fluency and the other aims of the 'Mathematics programmes of study: key stages 1 and 2 National curriculum in England (2014)', see page 45 in the 'Key mathematical ideas chapter'.

Maintaining children's fluency

Children's responses to mental mathematics questions and word problems, and their ability to make up their own problems in 'morning maths meetings' or in practice sessions, will inform you whether they are maintaining fluency with past learning.

You can also keep track of what children have remembered by choosing an activity from the practice suggestions of a completed activity group, varying the context of the problem and presenting it to children without preparing them in advance. Notice what they seem to have remembered and what they have not remembered. Plan accordingly when you reach the next activity group in that section.

In the whole-class and group-focus teaching sessions, choose some questions and activities from previous activity groups that will help to keep children's past learning 'simmering'.

Using the activity groups

The first page of each activity group is clearly coloured according to the **strand** it appears in (Getting Started – light blue, Pattern and Algebra – red, Numbers and the Number System – yellow, Calculating – dark blue). The title and the number of the activity group allow you to easily identify the content and how far through the strand you are.

The **key mathematical ideas** clearly highlight the important ideas children will be meeting within each activity group.

The **assessment opportunities** signal key information to 'look and listen' for, which indicates how much of the focus activities children have understood.

The **educational context** gives a clear outline of the content of the activity group as well as, e.g. how it builds on children's prior learning, how it relates to other activity groups and the foundation it establishes for children's future learning.

All activity groups have been extensively trialled in the classroom, so the **learning opportunities** come from real classroom experiences. They are designed to help children develop their understanding of the key ideas of an activity group.

Key mathematical ideas Fractions, Dividing, Equivalence, Mathematical thinking and reasoning — **Calculating**

Calculating fractions of amounts 14

Educational context

This activity group builds on children's understanding of fractions, developed in Numbers and the Number System 2, 6 and 7. It also revisits the method of using arrays introduced in the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, Calculating 11. Children calculate fractions of numbers and quantities, first to find one part, e.g. [illegible] of 48, then to find several parts, e.g. [illegible] of 48. They begin their exploration with arrays, before moving on to apply their understanding in real-life contexts, including money and measures. Finally, children build on the work in Calculating 9 and 13 to explore how to approach dividing calculations that produce remainders and how these remainders can be expressed.

Learning opportunities

- To use the inverse relationship between multiplying and dividing
- To use multiplying and dividing facts to find fractions of amounts
- To appreciate that having fluent recall of multiplying facts supports the process of calculating fractions of amounts

Words and terms for use in conversation

equal, fair, parts, divided into, dividing, multiplying, fraction, dividing facts, multiplying facts, factor, multiple, inverse, matching, how many … in … ?, dividend, divisor, quotient, remainder

Assessment opportunities

Look and listen for children who:

- Use the words and terms for use in conversation effectively.
- Find one part of a whole number or quantity.
- Find several parts of a whole number or quantity.
- Use multiplying facts to help with dividing.
- Describe part of an array as a fraction of the whole.
- Explain links between finding fractions and dividing.
- Express remainders in ways consistent with the context of the problem, including as remainders or fractions.

Explorer Progress Book 5c, pp. 12–13

After completing work on this activity group, give small focus groups of children their Explorer Progress Books and ask them to work through the challenges on the pages. As children complete the pages, assess what progress they are making with the central ideas from the activity group. Refer to the assessment opportunities for assistance.

Explore More Copymaster 28: Charity Funds

After completing work on Activity 3, give children Explore More Copymaster 28: Charity Funds to take home.

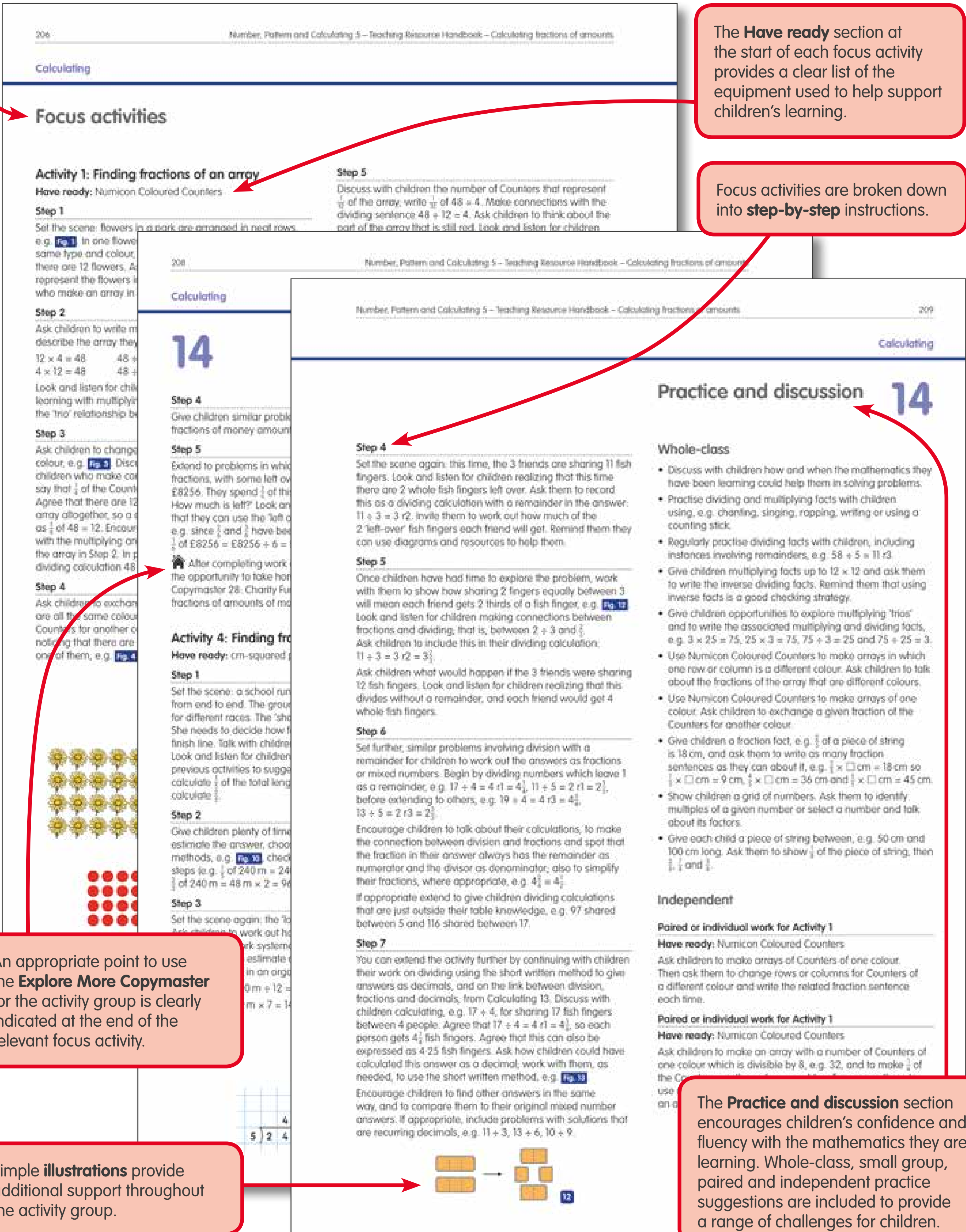
Explore More Copymasters provide an opportunity for children to practise the mathematics from the activity group outside the classroom through fun, engaging activities.

Clear links are made to the **Explorer Progress Books**. These books provide an invaluable chance to see children's thinking, monitor their progress and assess how much of the activity group they have understood.

Each activity group includes several **focus activities**, each clearly titled to show the specific learning points addressed. The first focus activity is a 'low threshold' activity allowing all children to engage with the work. Focus activities then build progressively to a 'high ceiling' that provides challenge and allows for differentiation within the same activity group.

The **Have ready** section at the start of each focus activity provides a clear list of the equipment used to help support children's learning.

Focus activities are broken down into **step-by-step** instructions.



An appropriate point to use the **Explore More Copymaster** for the activity group is clearly indicated at the end of the relevant focus activity.

Simple **illustrations** provide additional support throughout the activity group.

The **Practice and discussion** section encourages children's confidence and fluency with the mathematics they are learning. Whole-class, small group, paired and independent practice suggestions are included to provide a range of challenges for children.

Planning and assessment cycle

Here is a guide to show how planning can be informed by your assessments of children's understanding.

1. Choose an activity group	Before embarking on the activity group, review the educational context, consider your initial assessments of children and decide whether there is a need to revise any ideas children have met earlier.
2. Choose a focus activity	If this is the first lesson using the activity group, start with an early 'low-threshold' activity to include all children. Before the lesson, check the learning and assessment opportunities, brief teaching assistants and prepare resources.
3. Choose the practice activities	**Independent groups:** Refer to your assessment notes and allocate suitable practice activities (found at the end of the activity group). **Focus teaching groups:** Refer to your assessment notes and the learning and assessment opportunities from the activity group and allocate a focus activity.
4. Plenary session (normally during and at the end of lessons)	Think about the important ideas that children will have met in the lesson, particularly any generalizations that you want children to have made. Plan questions to prompt discussion and ask questions that encourage children to reflect on ideas they may have learned. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
5. After the lesson	Reflect on how children have responded in the lesson and note any significant steps on your assessment records. Use what you notice to determine the plan for the next lesson. Suggestions are given for whole-class practice that will help children to develop the ideas they have learned in the lesson. At some point after children have completed work on the activity group, ask them to complete the relevant pages of their Explorer Progress Book. This will allow you to assess how well they have retained the information they have been learning. It will also give you a chance to see how well they are able to apply this knowledge when faced with a 'non-routine' problem.

Creating short-term plans

Here is a template for how you might create a short-term plan. An editable version of this template can be found in the Planning and Assessment Support.

	Warm-up	Main Teaching Focus	Focused Group Work with the Class Teacher or Teaching Assistant	Independent Work	Plenary
Activity number/title	Select activities from the **Whole-class practice** section. This could be from a previous activity group to review and refresh children's previous learning.	Select one of the focus activities from the activity group matched to the needs of the children. Place the **activity number/title** of the chosen focus activity in your short-term plan.	Decide whether to: • select the next **activity number/ title** from the focus activities in the activity group. Place this in your short-term plan or: • consolidate the activity covered in the main teaching focus.	Decide whether to: • choose activities from the **Independent practice** section for groups, pairs or individual children. Make notes on your plan or work from the Teaching Resource Handbook or: • select a focus activity for groups to work on independently. Place the relevant **activity number/ title** in your short-term plan.	Encourage children to have a reflective conversation to draw together what has been learned in the lesson. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
Learning opportunities	Place the selected learning opportunity(ies) from the chosen activity group summary in your short-term plan.				
Notes and Educational context	Decide whether to: • use the activity directly from your *Number, Pattern and Calculating 5 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 5 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 5 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity. If working with a teaching assistant, you may want to select the relevant **educational context** from the chosen activity group.	Decide whether to: • use the practice or focus activity from your *Number, Pattern and Calculating 5 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	
Words and terms	Decide which **words and terms** you will use in **conversation**. Place these in your short-term plan.				
Resources	Prepare any resources you may need for the activity. Use the **have ready** section at the beginning of the focus and practice activities.				
Assessment opportunities	Select from the chosen activity group the **assessment opportunities** that you and the teaching assistant will be looking and listening for in the different parts of the lesson. Place these in your short-term plan. Remember to note whether children know when to use their understanding.				

How can I assess children's progress for teaching?

Assessing mathematics using Numicon involves making judgements about developments in children's mathematical communicating – both receptive and expressive.

As a result, you will need to know what the key developments are that you should look for. For this, you can check the assessment opportunities signalled in each activity group and consider how these achievements would show up in children's mathematical communicating. Specifically, you will need to look for developments in children's actions (what they do and notice), the imagery they use and respond to and their use of (and responses to) words and symbols in their conversation.

It is also important to notice children's fluency. For example, when is their communicating stilted, when is it punctuated by gaps and hesitations, and when does it flow consistently and well, suggesting a strong command of connections between well-established ideas?

This means that our approach to assessing *for teaching* does not involve giving children tests. Tests are mostly administrative devices for discriminating between children, to rank and select children and to allocate resources. Devised for administrative and comparative purposes, tests only sometimes accidentally reveal something immediately useful to a practising teacher.

Assessment should be as open as possible so that children can communicate as much as possible. It is through their mathematical communicating that you will gain a real insight into how children are thinking. This will enable you to make the most accurate assessment of their progress.

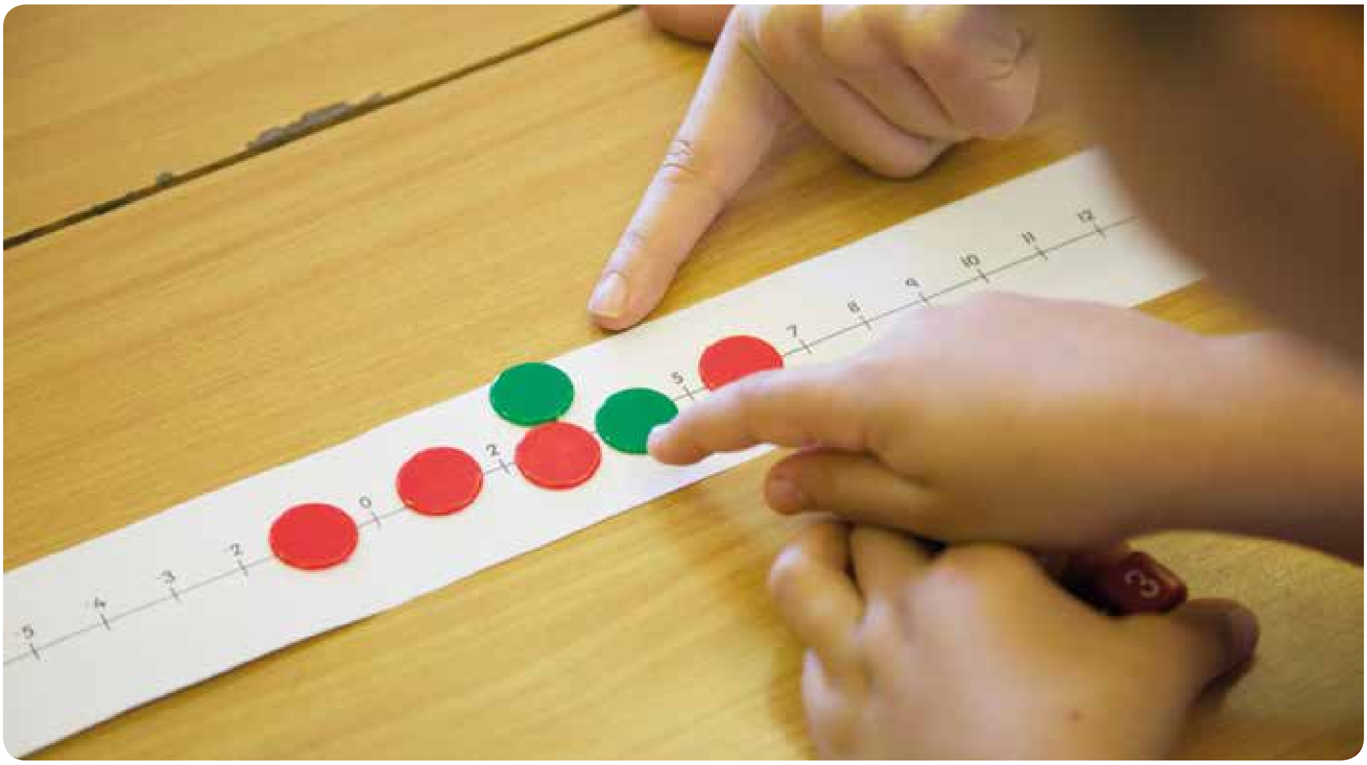
Specific challenges for the purposes of assessing are provided in the form of Explorer Progress Books (see page 9). Children cannot pass or fail at these assessment tasks; they simply respond in their individual ways. How they approach the tasks and their responses will inform you about their mathematical communicating and give you an opportunity to 'see' their thinking through the illustrating they use within the tasks. This level of insight into children's thinking will make it easier to gather meaningful and accurate assessments of where children are. Preparing for formal test situations is something quite different and is addressed on page 43.

Specific indications of children's progress

Each activity group lists several assessment opportunities that point to key achievements to look for during the work of that activity group. All of these achievements will be evident in children's actions, imagery and conversation as they progress.

Familiarize yourself with these well before you begin teaching any activity group; they will help to guide your interactions with children as they tackle the activities you give them. They will also indicate progress and provide useful information for grouping children and planning your teaching as you move on to the remaining activity groups.

If it helps, list the assessment opportunities on a sheet of paper against a list of the names of children you are teaching so that it is easy to note individual achievements and difficulties quickly as the work goes along.



Within each activity, there are also suggestions for what to 'look and listen for' as children are working on the activities. Focus on children's communicating and ask whether they know *how* to do the mathematics they are learning, and whether they know *when* to use it.

You will also find that how children use the Numicon Shapes, number lines, number rods and base-ten apparatus – and other resources – will give you insight into their thinking.

A child using materials by trial and error with muddled explanation would suggest they do not yet understand the activity. Plan to revisit it, focusing carefully on the mathematical language and imagery you will use. It may be that the child did not understand what they had to do.

Children self-correcting, i.e. working by trial and improvement, suggests their understanding is developing as they try out different solutions. Give children time to experiment and practise the activity and encourage discussion about their ideas.

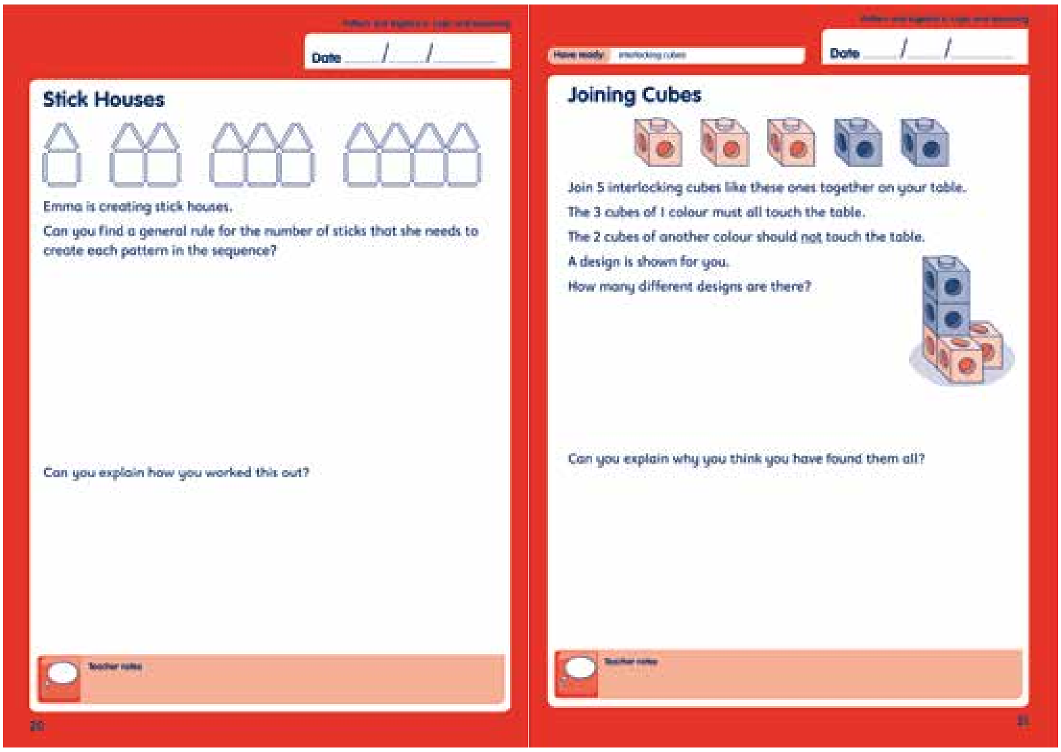
Children communicating clearly about what they have done (by talking, with apparatus and by writing it down), suggests solid understanding, so plan to move them on.

Assessing individual children's verbal counting

Reviewing children's counting regularly until they have secure understanding helps to keep track of how their understanding of the pattern of naming numbers in the number system is developing.

This review has to be carried out individually, but need not take long and can be part of the activities when teachers or teaching assistants are working with a focus group. At each review, first of all establish the child's counting range by asking, 'How far can you count?' If, for example, the child says they could count to 10 000, choose a number from within their counting range (9987) and ask them to count on (to 10 010). Then, choose another number (3303) and ask them to count back, stopping them when appropriate (e.g. 3287). Continue to choose numbers within their counting range for them to count on and back from, checking their pronunciation and their ability to count across multiples of 10, multiples of 100 and multiples of 1000. From time to time, continue to extend their counting range in whole-class sessions, modelling counting across multiples of 10, multiples of 100 and multiples of 1000. Also include counting in negative numbers and in fractions, e.g. counting on and back in quarters.

If you are concerned about other aspects of children's number understanding, refer back to the progression of the Numbers and the Number System activities in the *Number, Pattern and Calculating Teaching Resource Handbooks 4, 3, 2* and *1*.



What support is there for making summative assessments for teaching?

Assessment milestones and tracking children's progress

Within the medium-term plan for the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, you will notice that there are milestones (summary statements) of specific points that children need to be secure in before they can progress to the next section of activity groups.

The statements in each milestone are founded on the assessment opportunities in the preceding activity groups and are also aligned to the national curriculum in England (2014). Your ongoing assessment of each child will build up over the course of the activity groups and you can keep a record of each child's attainment, and track their progress against the collated milestones for the year, using the individual photocopy master in the Teaching Resource Handbook (photocopy master 1); or the milestone tracking support available in the Numicon Planning and Assessment Support.

At the point of each milestone, you can reflect on each child's achievements and decide whether you need to plan further support and practice for them, giving them time to consolidate their understanding, or whether they are ready to move on.

If children are moved on before they are ready, then their difficulties are likely to accumulate because they will not be adequately prepared for the new ideas they will meet.

Explorer Progress Books

Each activity group has two corresponding pages in one of the Explorer Progress Books. The tasks in the Explorer Progress Books have been designed to present children with tasks that give them opportunities to use the mathematics of the activity group. One page generally poses a problem that challenges children to use the mathematics they have been learning in the activity group within a new context. The other page generally aims to provide more open opportunities for children, enabling you to assess their ability to think mathematically more widely and also allowing you the opportunity to see the methods children use as they persist with an exploration.

The tasks are not tests, and so are as open as possible, inviting a full range of responses. Children should have available to them all the imagery and materials that have been available during the teaching of the activity group, and should be invited to express what they are doing as they do it. It is best to avoid affirming or denying anything a child says or does as they work. Instead, look and listen for what children do without your guidance.

In order to help you assess whether a child knows *when* to use the mathematics of the related activity group, the Explorer Progress Book tasks are designed to reveal as much as possible about the breadth of a child's understanding, so that you know what needs to be addressed in the future. These are not designed to be pass/fail tests, but are there to support you in assessing as accurately as possible children's current and/or retained understanding.

Give careful consideration to when children should be given each Explorer Progress Book task. You might ask them to complete one page at the end of work on the activity group, and then another two weeks later to see how much seems to be being retained.

You might give children both tasks after the following activity group. You might give these tasks just before children face the next connected activity group. The point of these tasks is to discover children's understanding at a particular point – decide when the most useful point would be, in each case.

It may also be useful to keep notes about children's responses, and what you see as their significance for future work.

The Explorer Progress Books are designed to be given out to small focus groups, so that you can administer and monitor each individual child's responses to the pages. In this way, you will be able to build up a developing idea of each child's progress during the course of the school year.

What about formal testing – for national authorities?

Formal tests and examinations are important hurdles for children and teachers, parents and carers, schools, universities, professionals, employers and governments. However, the nature of a formal test means that it tends to be an artificial and unique setting in which to 'do mathematics' and, in this sense, does not correspond with how children encounter mathematics in their mathematics lessons or everyday lives.

As a result, preparing for national tests needs time devoted to children's preparation for the uniqueness of the experience. Doing mathematics in the circumstances of a formal test or examination, however, should not become the paradigm for 'doing mathematics'. Children need to learn to function mathematically in a very wide range of situations.

Therefore, it is important not to confuse formal testing with 'doing mathematics' in any other situation. In a formal mathematics test, communicating is almost always severely restricted to written forms only; this allows for some imagery, but not usually for action with physical materials. Also, the written language common to mathematics test papers can be very formal. Children will need plenty of prior practice at interpreting such writing.

Children will also need plenty of practice at 'internalizing' their use of action and imagery, since physical models and screen-based imagery will not usually be available to them during a test. When using Numicon, there are many opportunities to encourage development of children's mental imagery, and children should be continually encouraged to 'imagine' actions, objects, movements and shapes as well as working physically, as often as possible.

Finally, think about how children will need to react when they encounter 'difficulty'. In their mathematics lessons, children will have been encouraged to express difficulty, to explain why something is challenging, and to use action and imagery to illustrate their thinking. Under exam conditions, children will have to communicate mathematically with themselves, work hard to express silently what the trouble is, use mental imagery, and thus respond positively to being 'stuck' in an exam.

Formal examinations and testing are a fact of life. They are important and children need to prepare for these unique events. However, it is also important to recognize that examinations are, by their nature, artificial and not representative of what 'doing mathematics' is about in any other situation.

In order to make ongoing assessments of children's understanding, allow them the full range of actions, imagery and conversation, and encourage them to communicate mathematically.

Key mathematical ideas

Underlying the activities in Number, Pattern and Calculating 5 are many key mathematical ideas that children will be developing and extending, as well as some conventions they may be meeting for the first time.

In order to teach these ideas and conventions effectively, those who are working on activities with children will need to be very clear themselves about the mathematical content involved in each activity.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the activity groups of the *Number, Pattern and Calculating 5 Teaching Resource Handbook*. Exploration of the mathematical ideas within this section will help with planning how to develop these ideas with children.

The introductory page for each activity group lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to the following section.

The mathematics coordinator may also find it useful to work on these key mathematical ideas in professional development sessions with the class teachers and the wider school staff.

In this section you will find overviews of:

Fluency, reasoning and problem solving – the aims of the 'Mathematics programmes of study: key stages 1 and 2 National curriculum in England 2014'

Mathematics is an activity, and 'doing mathematics' essentially involves communicating and thinking in distinctive ways. Numicon activities are devoted to supporting children's development of their mathematical thinking and communicating – both with others and with themselves – through inviting them continually to be active, to illustrate and to talk.

Fluency is an attribute of communicating to do with an effective smoothness of flow; it should not be confused with being able to do something quickly. Understanding is involved in fluency, and a simple and mechanical calculating proficiency alone is not enough for children to become fluent in doing mathematics.

Numicon activities develop richness in children's thinking and communicating by encouraging them to be active, to illustrate their thinking and to talk. Such richness in communicating helps children to respond flexibly to challenging situations, and encourages them to develop the habit of always looking for a variety of ways of communicating about (and hence thinking about) the structures and possibilities of new situations. Flexibility is what supports mathematical fluency in new and unfamiliar situations.

'Automaticity' necessarily supports fluency as children develop and progress to facing ever more complex situations. As every learner driver understands very well, many actions and responses need to become automatic in order to drive a vehicle smoothly and effectively in complex traffic situations. So it is with doing mathematics; the more automatic certain responses become, the more able we are to direct available attention to reflecting and acting on the more complex aspects of problem situations. With Numicon, the practice necessary to developing familiarity and automaticity is integral to all activity groups.

Flexibility and automaticity together are what allow children quickly to assess any calculation as either 'One that I could do easily this way', or as 'One I could change into something much easier'. In all the open discussion encouraged

throughout Numicon activities, in work on 'non-computational thinking' and in the explicit emphasis upon algebraic relations that underpin effective calculating, children are encouraged to approach calculating in a *thoughtfully* fluent manner, rather than mechanically. This is fluency based upon understanding, not upon mechanical repetition.

Although many people assume **reasoning** to be about using only words and symbols logically, in mathematics, reasoning is much wider than that. Imagery is involved in almost all mathematical thinking and reasoning, and so is action. The action, imagery, conversation and relationships that are at the heart of Numicon constitute all the essential elements that will, together, develop into children's mathematical reasoning. For example, in the Number, Pattern and Calculating 5 activities, reasoning about sequences of 'square' or 'cube' numbers, or even about some numbers being 'bigger' than others, will involve much action with imagery.

Even though all that might be on a child's page are written numerals and other symbols, the communicating the child has done or is doing with themselves (i.e. their thinking) is bound to involve action, imagery and words as well. Reasoning about abstract number ideas depends upon visual imagery such as number lines and patterns to communicate the 'logic' of the interrelationships involved.

Being able to reason by following a line of enquiry, conjecturing relationships and generalizations, and developing an argument, justification or proof using the full range of mathematical communication are integral to mathematics teaching with Numicon.

Problem solving, by which we mean addressing a mathematical challenge that the solver doesn't immediately know how to approach, will involve reasoning, thinking flexibly, imagination, and also courage – the kind of courage that grows in situations where children learn that challenge is normal, imagination is praised, and that perseverance is generally seen to pay off. Using Numicon, there are three key things that help children develop the qualities of imagination, courage and persistence.

Firstly, to support children's approaches to problem solving, the mathematics they are learning is grounded within contexts in which it is seen to be useful. If children can 'see the point' of the mathematics they are being asked to learn, they are a good way towards knowing when that mathematics would also be useful at other times.

Secondly, there is an acknowledgement that doing mathematics is challenging – unless children are in a familiar situation that they have recognized. It is expected that children will find most Numicon activities non-routine and suitably challenging, and that challenge will become normalized. When children are challenged – and especially when they are 'stuck' – it helps them enormously to try to communicate the difficulty they are experiencing as fully and richly as possible.

This is where the rich variety of communicating that is continually encouraged with Numicon activities comes into its own; as children try to communicate a difficulty in as many ways as they can, new ways forward will almost always occur to them. The central message to children is that when they don't know what to do, they should aim to communicate mathematically, and express their difficulty as precisely as possible.

Thirdly, in the Explorer Progress Books, children are regularly supplied with unfamiliar and quite open situations that invite them to use the mathematics they have been learning. These experiences are designed both to confirm the message that challenge is normal, and to identify particular aspects of their work that will benefit from further attention.

By facing mathematical challenge, and by developing their resilience and resourcefulness in the face of such challenge, children are learning to solve both routine and non-routine problems as independently as possible. Problem solving can then become a familiar and enjoyable experience.

Communicating and mathematical thinking

Numicon activities are aimed at developing children's mathematical thinking and communicating. Children learn to think mathematically by learning to join in with the ways we, as expert mathematical thinkers, communicate. Children's thinking is their own developing version of the mathematical communicating they meet around them, both within classrooms and in the wider world.

Mathematical communicating and thinking is distinctive in several ways; mathematics is essentially about looking for

patterns and regularities in situations, generalizing about them, and thus gaining control within a context, whether it involves shapes, quantities, weather forecasting or relationships of any kind. Because this is what we do in thinking and communicating mathematically, mathematical communicating and thinking typically display the following key features:

1. **Generalizing**: In doing mathematics we are constantly generalizing, and then generalizing about our generalizations. In our early years we might generalize that every time we add five objects to four objects, *whatever they are* we always end up with nine objects altogether, and we learn to communicate this generalization conventionally in mathematics by writing and saying:

$$4 + 5 = 9$$

Later on, as we get more used to working with numbers, we start to make new generalizations about these early generalizations, such as: 'Whichever way round you add two numbers, they always come to the same total' – and we would write and say this conventionally in mathematics as:

$$a + b = b + a$$

This new equation[1] uses letters because we want to make a general statement about adding *any* two numbers, and we cannot say or think this by using particular numbers – it wouldn't be a general statement if we did. We use 'a's and 'b's in our thinking and communicating here because we are generalizing about numbers, and this generalizing using letters is a key feature giving mathematical communication a distinctive style – we frequently use few words, and are concise with symbols.

As another example, in the activities in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* children are introduced to communicating explicitly and mathematically about 'distributivity', which is a property that explains in general how we can break down a 'long multiplication' into smaller steps. We can communicate this concisely and mathematically however by using the following very short identity in which again a, b and c stand for *any* numbers:

$$a\,(b + c) = ab + ac$$

For many people, this signals a move into something called 'algebra', but what is actually different is that we have moved up a level of generalization – we are now generalizing about our first generalizations (i.e. about our first numbers) – and this requires a new form of communicating, with letters; notice that if you change the thinking (e.g. in generalizing), you change the communicating (in this case by moving from numerals to letters), and vice versa.

In Number, Pattern and Calculating 5, children are also increasingly invited to think about generalizations more explicitly, and to learn to recognize and use different ways in which general statements can be made. Examples might include: 'All prime numbers have …', or 'A prime number has …,' or 'Any prime number has …' Children will need to discuss such statements very carefully in order to be clear about what is being claimed, and also to understand the distinctive ways in which we communicate these things mathematically; by looking after children's communicating, we look after their thinking.

2. **Being systematic**: As we make generalizations in mathematics, we often also make new categories: for example, 'Prime numbers are whole numbers with exactly two different factors'. Then, if we want to find out whether 163 is in that category, for example, we will have to check out all its possible factors before deciding whether it is a prime number or not, and we will have to do this systematically in order to be sure we have looked at *all* the possibilities. When trying to decide in mathematics whether something is possible, or true, or not, we need to make ourselves sure by finding a way of exploring all possibilities *systematically*. (Some useful tests of divisibility that could help to determine whether 163 is prime are also introduced in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 3.)

3. **Being logical**: Our generalizations are also crucially involved in our logical reasoning as we do mathematics. A great deal of mathematical reasoning involves following up implications, which are usually expressed as, 'If this is true, then that is also true'. For example, 'If it is impossible to make a rectangular array of more than one row with 11 counters, then 11 is a prime number'. (And we could find this out by being actively systematic – using 11 counters, we can physically try all the possibilities.)

1 This type of equation, which involves values that vary ('a', 'b', and 'c') and yet is always true whatever values the variables take, is called an 'identity'.

This implication follows from generalizations that, 'All composite numbers of counters can be arranged into one or more rectangular arrays involving at least two rows' and, 'All whole numbers are either composite or prime numbers' and, 'No whole number can be both composite and prime'.

Such reasoning sounds very formal but, as children work on the Number, Pattern and Calculating 5 activities, they should be encouraged to follow through 'If … then …' implications in their reasoning wherever possible, and to discuss such implications fully whenever they arise. (Visual and physical illustrations are often particularly helpful in discussions like this, as with all mathematical thinking and communicating.)

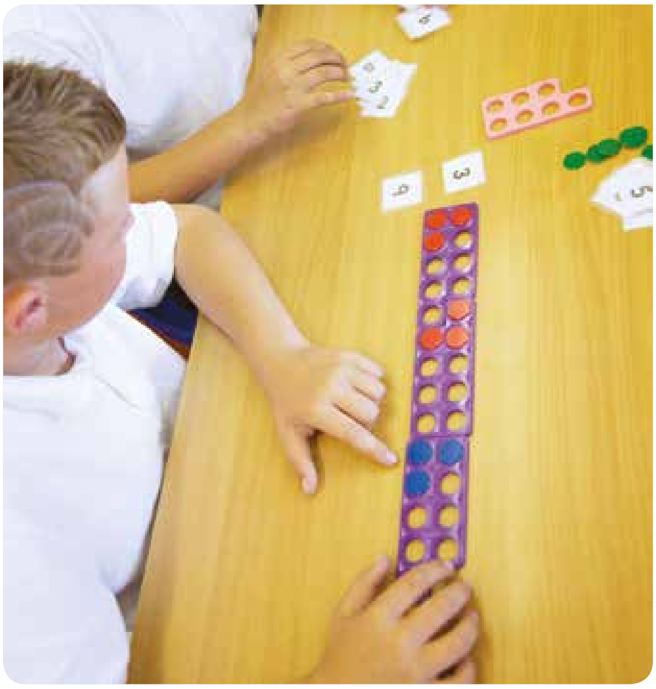
4. **Contexts**: As children learn to use numbers in an ever wider range of situations, and as they meet new and different kinds of numbers, so they will at times find themselves thinking and communicating mathematically in two quite different kinds of context: in so-called 'real-world' contexts and in mathematical contexts.

In a real-world context, the thinking and communicating is about real-world objects and relationships; in a mathematical context the communicating and thinking is about mathematical objects, e.g. numbers. In Number, Pattern and Calculating 5, children will find themselves moving often into mathematical contexts as they meet and discuss new kinds of numbers and as they move closer to formal algebra.

Children will continue to find connections signalled between the mathematics they are doing at any stage and real-world contexts in which that mathematics can be useful. However, they will also spend much more time now in the Number, Pattern and Calculating 5 activities working simply on new kinds of numbers and how these numbers relate to each other; these are times when they will be working within a mathematical context on mathematical objects – pure numbers are mathematical objects.

For example, as children learn to treat fractions as objects, i.e. learn to talk about '$\frac{3}{5}$' as a thing and not as '$\frac{3}{5}$ *of*' something, as they learn to convert $\frac{3}{5}$ to 0·6 as a decimal fraction and to 60% as a percentage, as they learn to interpret $\frac{3}{5}$ as 3 ÷ 5, and as they learn to calculate more generally with fractions, they will be thinking and communicating strictly within a mathematical context, about mathematical objects. This can be difficult, and many children will often feel such work is too 'abstract'.

It will help during such work, then, to encourage children to move between real-world and mathematical contexts, and to use as many different kinds of visual and physical illustration as possible, including different materials, drawings and so on. Successful mathematical thinking and communicating at the level of Number, Pattern and Calculating 5 involves being able to move smoothly between real-world contexts and mathematical contexts, to move smoothly from *general* (mathematical) relationships to *particular* (real-world) cases, as well as to move in the opposite direction from seeing a pattern in particular (real-world) cases to making a (mathematical) generalization.

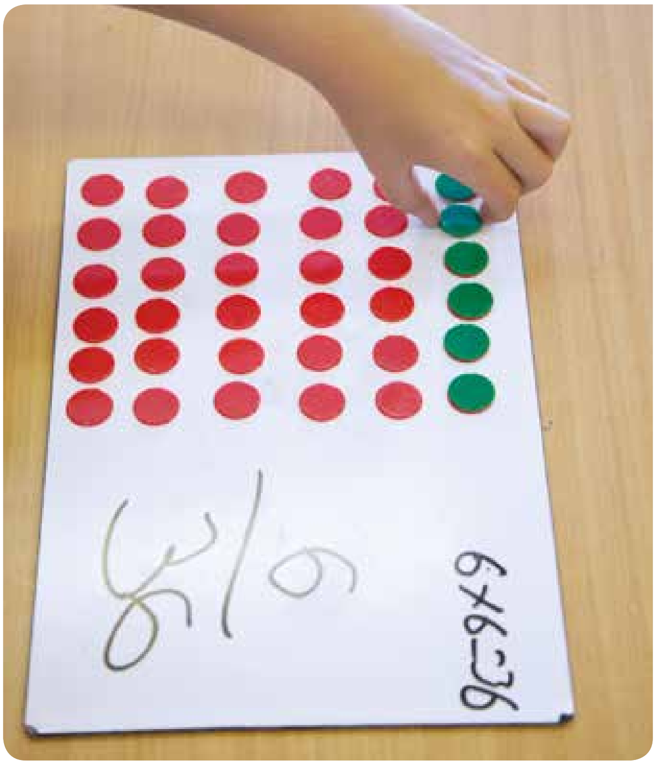


So, many children who balk at working out an abstract 7·6 – 2·75 = □, for example, will be able to subtract £2·75 from £7·60 perfectly accurately; these are children who are at the moment more comfortable with real-world objects than with mathematical ones. Exploring a wide range of illustrations will generally help children to begin to connect the two types of object and the two types of context. Number lines, for example, are excellent illustrations in a situation like this; since physical distances along a line represent abstract numbers, mathematical objects here are directly represented by something physical. Try asking them about where 7·6 and 2·75 appear on a number line and 'seeing' their difference visually.

A great deal of mathematical thinking and communicating thus involves learning how to move between real-world and mathematical contexts. In Number, Pattern and Calculating 5, children are constantly engaged in this movement between these different worlds as they use many more kinds of mathematical object than in their earlier mathematics; encourage them to illustrate and to talk about such to-ing and fro-ing between 'real' and mathematical objects wherever possible.

In summary, there are four key aspects to communicating and thinking mathematically:

- **Generalizing:** Our mathematics works through us spotting patterns in situations and generalizing about them. Our mathematical communicating and thinking is consequently characterized by a predominance of generalizations, and by generalizations being carefully related to each other.
- **Being systematic:** In order to reach a generalization, or to question whether a given statement is true, we often have to be systematic in order to be sure that we have investigated all the possibilities in a situation. For example,



'You can pay cash for a 10p item in exactly 11 different ways' would have to be tested by exploring all possible combinations of coins *systematically*, in order to be sure that we have actually considered them all.

- **Being logical:** We reason with our generalizations, for example, 'If multiplying two odd numbers together never produces an even number, then 59 x 23 *cannot* equal 1356'. Children need much careful experience in following through such *implications* of generalizations and of other kinds of statements.
- **Contexts:** Children need to learn *when* to use any particular piece of mathematics, not just *how* to do it. It is no use being able to rehearse all your times tables facts perfectly if you don't know when to multiply or divide. Constant to-ing and fro-ing between everyday situations and the mathematical world of 'abstract' generalizations helps children get to know when doing some particular mathematics is helpful. This is an essential element in children learning to solve mathematical problems successfully.

Pattern and algebra: essential to mathematics for children of all ages

An essential idea underlying all Numicon activities is that of pattern. Pattern may not sound like a particularly mathematical idea, as we are used to patterns of one sort or another occurring in so many non-mathematical contexts. We could not have learned to speak, for instance, without noticing patterns in the sounds we heard as infants; patterns (rhythms) structure music and dance, and patterns in stories and plays enable us to anticipate the unfolding of a plot (they also, of course, allow our expectations to be manipulated by writers). Much poetry depends upon patterns for its effect, and most scientific research is an attempt to discover or establish patterns in observable phenomena.

Importantly, it is the detection of patterns in our experiences that makes essential aspects of our lives predictable. And since seeing patterns is absolutely vital to human survival, this is also something humans generally do well. Seeing patterns is what enables us to generalize and then to predict what comes next, thus gaining a measure of control over our environment and futures.

Patterns are essential in mathematics for a very special reason: they enable us to imagine actions, events and sequences going on 'forever' without us having physically to work out and wait for each and every step. It is patterns that allow us to generalize into the future. Counting is a good example. As we have invented a system for generating number names, we can imagine what it would be to count forever *without ever actually having to do it*. Most people know they could count to one million, without ever having done it. They know they could because they know the place value system; they know the patterns in number names that would enable them to go on forever were they ever called upon actually to do so. Importantly, once children see the pattern that each next whole number is 'one more' than the previous one, they also know how counting things may go on forever – a vital generalization (called the 'successor relation') that allows children to work with collections of any size.

As another example, by noticing the pattern that it doesn't matter which two numbers we choose to multiply first when we are multiplying three numbers together (a generalization we call the 'associative property'), children gain an important flexibility in calculating that allows them to find quick ways to multiply mentally. For example, 48×250 can become $(4 \times 12) \times 250$, which then becomes $12 \times (4 \times 250)$, which is then easy to calculate as 12 000. And because they have generalized in this way, they don't have to keep checking every example.

It is impossible to overestimate the importance of pattern to mathematical thinking. In fact, a very large part of algebra, often thought of as the most powerful branch of mathematics, consists of seeing, manipulating and generalizing from patterns. It is algebra that enables humans to launch space shuttles and bring them back successfully, by predicting and generalizing from patterns. It is important to remember that in all the key mathematical ideas discussed here, pattern and generalization are fundamental.

A note on formal algebra: There is a popular view that 'algebra' begins when we start using letters instead of numbers. It is more helpful, however, to think of algebra as the study of relationships *per se*, and in this sense doing algebra with numbers is not concerned with particular numbers as such, but much more with relationships between them.

We use letters as we do formal algebra for two main, but different, reasons: firstly when we want to do some calculating involving an unknown quantity, and secondly when we want to work on relationships that apply to whole (sometimes infinite) ranges of numbers (e.g. in formulae, such as $A = \pi r^2$). In neither case can we do our communicating just with particular numbers, hence we use letters to 'stand for' unknown amounts, and/or for varying values.

Children learn about and use relationships between numbers (e.g. **equivalence**) from early on in their schooling, and by the time they are six years old (in the UK) they will be using an 'empty box' notation to represent unknown amounts, for example in number sentences such as:

$$23 + \square = 107$$

Thus in an important sense children actually begin doing 'algebra' from quite early on in their schooling, as they look for patterns (generalizing), as they study relationships between numbers (e.g. **equivalence**, **inverse**), and as they use such number relationships to work out unknown amounts represented by 'empty boxes' (instead of letters). Although many think that 'algebra' begins in secondary school, in a very real and important sense children have been doing algebra since they first began to notice, think and talk about relationships; secondary school algebra is just a formal version.

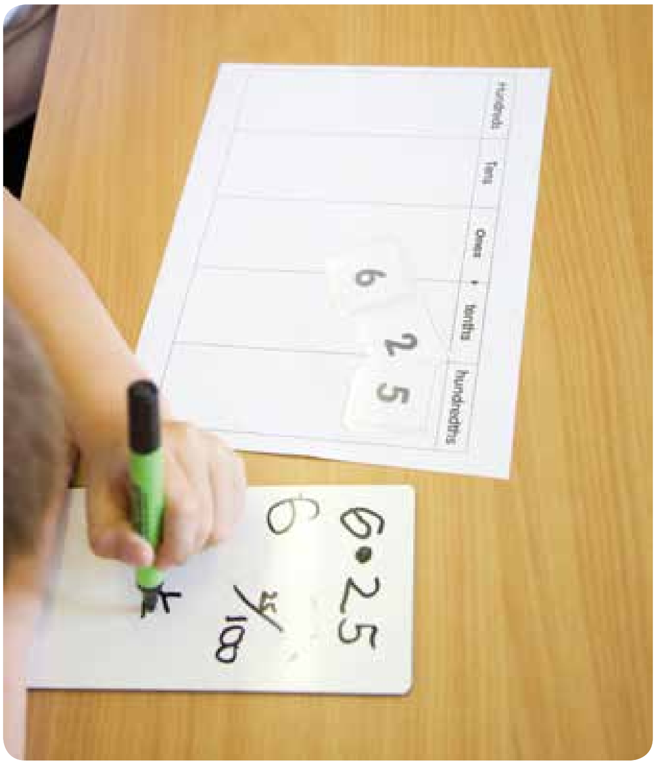
Patterns in using the four operations: important algebraic relationships

As children begin to calculate in practice, we can help them to notice particular patterns (or connections) and then to generalize from what they have noticed. Usually we do this without deciding to give these features a formal mathematical name. For instance, we can help children to notice that it doesn't matter which way round we do any multiplying as we will always get the same answer, and then invite them to use that observation to calculate 4 × 7 if they can't remember 7 × 4. We may do that in practice, but we don't often call it 'the commutative property of multiplying', although we might say 4 × 7 and 7 × 4 are equivalent.

Communicating and thinking about how arithmetic operations work together involves important algebraic ideas – they are about number relationships – and consequently within mathematics these ideas do have formal names. Whether or not we think it important that children know and use these formal mathematical names, it is important for children's mastery of calculating that they understand what are called the 'properties' of and relationships between the four operations with numbers, especially as they become older and approach formal algebra. In what follows, we use the formal mathematical names for those properties and algebraic relationships that are important at this stage.

Equivalence

Equivalence is one of the most important mathematical relationships of all and yet it is often the case that not



enough attention is paid to it explicitly as we discuss work with children. Children often work with equivalence implicitly from very early on in their thinking, but in doing mathematics at this stage we definitely need to discuss instances of this relationship fully, and allow children plenty of time to reflect.

We signal an equivalence relationship in mathematics by using the symbol '=', for example by writing:

$$\tfrac{4}{5} = 0{\cdot}8 \text{ and } 3 + 4 = 7$$

Equivalence literally means 'equal value'. Quite often the most interesting and important instances of equivalence occur when two or more things are of equal value but look different. In early calculating, there are three occasions when children face important instances of equivalence: the introduction of the '=' symbol itself (which means 'is equivalent to'), when they encounter quantity value and column value (see the section on **names for numbers: counting numbers and place value**), and when they meet fractions and percentages ($\tfrac{1}{2} = \tfrac{2}{4} = \tfrac{3}{6} = 0{\cdot}5 = 50\% = 1 \div 2 \ldots$). However, there are numerous other occasions when children need to see equivalence and we are often less explicit.

For their mental calculating strategies to make sense, children have to be able to see what are called the 'decompositions' of any number as equivalent to each other, for example 9 = 1 + 8 = 3 + 6 = 10 – 1. Similarly, factor pairs are all equivalent, e.g. 1 × 16 = 2 × 8 = 4 × 4 = 8 × 2 = 16 × 1. In measuring, equivalences between units (100 cm = 1 m, 1″ = 2·54 cm – activities on this are included in the *Geometry, Measurement and Statistics 5 Teaching Resource Handbook,* Measurement 1) are at the heart of being able to understand and relate systems of measurement.

When young children seem not to understand something that is clear to us in calculating, there is often an equivalence we see that they do not, or vice versa. We might fully realize that 4 ÷ 5 = $\frac{4}{5}$ for example, before children have come to equate the *process* of dividing with fractions that have become 'objects' to them. Don't forget that equivalence is about things that are worth the same, but which look different; very often we find it difficult to see past appearances. We tell children that 'tens' are not 'ones' even though their numerals look the same, and yet we also tell them that tens and ones are 'the same thing' (equivalent) when it comes to breaking up a ten to make ten ones.

There is often also a language problem here. As it may sometimes seem unhelpful to use the formal word 'equivalence' with children, we can often resort to an easier word, 'same', when talking about equivalence; we often say, 'It's the same thing.' Unfortunately, equivalence does not mean 'the same thing' – it means **equal value**, **different appearance**.

In Number, Pattern and Calculating 5, important equivalences between expressions are further explored – equivalence between common fractions and decimal fractions, between equivalent proper fractions, between mixed numbers and improper fractions, and between all of these and percentages. It is important to children's developing understanding that they realize that 0·5 and $\frac{1}{2}$ and 50% and '1 ÷ 2' are all different, and yet equivalent, ways of referring to the same value (see the section on **fractions**).

Inverse relationships

Adding and subtracting have what is called an 'inverse relation' to each other. What this means is that each can undo the other. If I add 6 to a number, I can then undo that adding by subtracting 6, and vice versa. This knowledge is important to children for several reasons. Firstly, the more connections children can make between things they learn, the more meaningful their learning is. Secondly, it is important that children don't think adding and subtracting are completely unconnected, because if they do they will never understand the 'inverse of adding' structure of subtracting. Finally, children should understand that adding or subtracting can always be checked by doing the inverse calculation: we can always check our adding by subtracting, and vice versa.

Multiplying and dividing also have an inverse relation to each other. Noticing how dividing undoes multiplying (and vice versa) is crucial to connecting these two operations with each other. This will also help children to see that, if multiplying is seen as repeated adding, it makes sense that dividing can be seen as repeated subtracting (quotition). These are important foundations for the extended multiplying and dividing calculations children will explore in Number, Pattern and Calculating 5.

As is noted on pages 55–56 in relation to counting and place value, 'partitioning' numbers in various ways while calculating, is the inverse of the 'grouping' objects into tens action that children have done earlier in answering 'how many?' questions without counting.

Inverse relationships are also used in relation to the 'empty box' notation that was first introduced in the *Number, Pattern and Calculating 1 Teaching Resource Handbook*. Children begin to learn the important algebraic use of symbols to stand for unknown amounts by using empty boxes. For instance, in asking them to solve 3 + ☐ = 10, we ask them to work out which number should go in the box to make the number sentence true. This type of apparent 'adding problem' requires children either to 'undo' a number fact they can remember, or to subtract 3 from 10; in either case they are using an inverse relationship. Empty box problems also ask of children a clear understanding of equivalence.

There is an increased emphasis upon inverse relationships, as children are invited to 'work backwards' and to 'undo' mixed calculations in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 2. This is not only preparing children for a later idea of 'inverse functions', but contributes importantly to the development of children's non-computational thinking.

Although we do not yet make this explicit to children, from the *Number, Pattern and Calculating 4 Teaching Resource Handbook* onwards they have also met inverses of another kind. In negative numbers, children are meeting what are called the 'additive inverses' of positive numbers. Put simply, $^{-}2$ is the additive inverse of $^{+}2$, $^{-}3$ is the additive inverse of $^{+}3$, and so on. If we add any two 'additive inverses' together, the result will always be 0 (zero); in an important sense, an additive inverse undoes what its partner can do – and together they are equivalent to 'doing nothing'.

In unit fractions, children are meeting the 'multiplicative inverses' of whole numbers, e.g. $\frac{1}{2}$ is the multiplicative inverse of 2, $\frac{1}{3}$ is the multiplicative inverse of 3, and so on. If we multiply any two multiplicative inverses together, the result will always be 1. Any multiplicative inverse undoes what its partner can do: together they are equivalent to 'doing nothing' when multiplying, that is, equivalent to multiplying by 1.

It is not expected that children working through the Number, Pattern and Calculating 5 activities will be discussing multiplicative inverses with you or with each other. What is important is having an awareness of how children are gradually learning more and more about the individual roles of numbers and operations that go together to make up what is called our 'real number system'. It is crucial that children increasingly understand how all the individual pieces – new operations, new kinds of numbers – fit together into this coherent system, and inverse relationships are an important part of that.

Zero and one: examples of 'doing nothing'

Most children notice that there's something funny about zero. Quite rightly, too: there is. Within adding and subtracting, zero is what is called an 'identity element', which means that operating with it leaves everything exactly as it was; adding or taking away zero amounts in effect to doing nothing. Children need plenty of help understanding this because (again, quite rightly) they can't see the point of doing nothing. There is no point; it is simply that zero is a number – it has its own important position on the number line and it can be added and subtracted. It just gives a strange result when added or subtracted: no change at all.

Notice that, when multiplying or dividing, 1 is the identity element; multiplying or dividing by 1 leaves everything just as it was. In contrast, as multiplying and dividing are introduced, the role of zero becomes even more bizarre. In fact, in these operations zero becomes a kind of rogue element, destroying everything it touches. Children will find that multiplying anything by zero always results in zero itself – a very strange result. Even stranger is the fact that dividing by zero is simply not defined in mathematics; it is a calculation with no answer at all. Not many children ask about dividing by zero, although by the time they are working on the Number, Pattern and Calculating 5 activities, some may begin to do so; if you are asked, try talking through and exploring with children, 'What would happen if we tried?'

Commutative property

Adding has what is called a 'commutative property'; subtracting does not. It does not matter which way round you do an adding sum; it does matter when you are subtracting: 12 + 6 equals 6 + 12, but 12 – 6 does not equal 6 – 12. Similarly, multiplying (because of its repeated adding structure) is commutative; dividing (because of its repeated subtracting structure) is not.

Associative property

If you have three numbers to add together, it doesn't matter which pair you add first before then adding the third. With 2 + 3 + 5 for example, you can add the 2 and the 3 first, or the 2 and the 5 first, or the 3 and the 5 first. Whatever you do, you always get the same answer: 10. Because of this, adding is said to have an 'associative property'. The same applies to multiplying: try 2 × 3 × 5.

With subtracting this doesn't work. Try it out with 12 – 4 – 1. Is the answer 7 or 9? It could be either; we don't know. Examples like this explain why we need to use brackets in many numerical and algebraic expressions; in this case, using brackets makes the expression clear and unambiguous: (12 – 4) – 1 = 7 and 12 – (4 – 1) = 9.

Similarly, division does not have an associative property. Try 24 ÷ 2 ÷ 3. Is the answer 4 or 36? Quite a difference! Now try (24 ÷ 2) ÷ 3 and compare your answer with 24 ÷ (2 ÷ 3). Once again, using brackets is necessary to make the original expression clear and unambiguous.

In the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, Pattern and Algebra 3 children were introduced to the use of brackets, in order to be clear about expressions such as 6 × 5 – 2 × 5: recording this as (6 × 5) – (2 × 5) so that children do not work out a value for the whole expression by carrying out the series of operations in the order they read them, from left to right. From Number, Pattern and Calculating 4 onwards, children will learn to follow the convention of (in a composite calculation) working out the value of whatever is inside brackets first.

Distributive property

In Number, Pattern and Calculating 5, children continue to make use of a very important relationship between adding

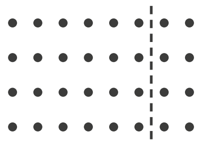
and multiplying that helps us to multiply large numbers together quite easily. To give it its full name, 'the distributive property of multiplication over addition' is what allows us to break down a large multiplying calculation into a series of smaller calculations that are easier to do mentally.

Expressed algebraically, this property may be summed up as:

$$a\,(b + c) = ab + ac$$

which really means, 'You can do any multiplication a bit at a time, and then add up the individual results to get the final answer.'

The key image that we use to illustrate this property is that of an array. For example:



This array can be used to illustrate not only that 8 × 4 = 4 × 8 (commutative property), but also that

$$4 \times (6 + 2) = (4 \times 6) + (4 \times 2)$$

This is the property that lies behind both traditional written methods of 'long multiplication' and the 'grid method' of multiplication, and of course applies to all numbers, however large. Thus we can break down multiplying 247 by 7 into three more manageable chunks:

$$\begin{aligned} 7 \times (200 + 40 + 7) &= (7 \times 200) + (7 \times 40) + (7 \times 7) \\ &= 1400 + 280 + 49 \\ &= 1729 \end{aligned}$$

and this is exactly what happens within the written column method of multiplying that children learn at school.

Non-computational thinking

All of the patterns and relationships described so far are used in what has in recent times become called 'non-computational thinking', and this important work continues to be developed in Number, Pattern and Calculating 5. 'Non-computational thinking' is a term that has begun to be used to describe ways of manipulating relationships between numbers *without* actually calculating or computing a numerical outcome.

Non-computational thinking is important for at least two reasons: firstly, it helps to lay a foundation for children's later formal algebraic thinking; secondly, it is often extremely useful for converting an apparently complex or difficult calculation into an equivalent and easier one.

As children move towards formal algebra, they move towards describing relationships between both known and unknown numbers, explicitly and symbolically. As an example of something children will need to be able to work with later on, think about the equation $2x^2 - 7x + 3 = 0$, in which x represents an unknown number, or possibly, more than one number.

As children work to solve this equation, that is, to work out what values x might have, they will need to manipulate this combination of known *and unknown* numbers in exactly the same way that we calculate with all known numbers. You may remember that children will need to 'factorize' this equation, i.e. work out a pair of factors that, multiplied together, produce the expression on the left of the equation. Such a pair of factors turns out to be $(2x - 1)$ and $(x - 3)$, so that we can then say:

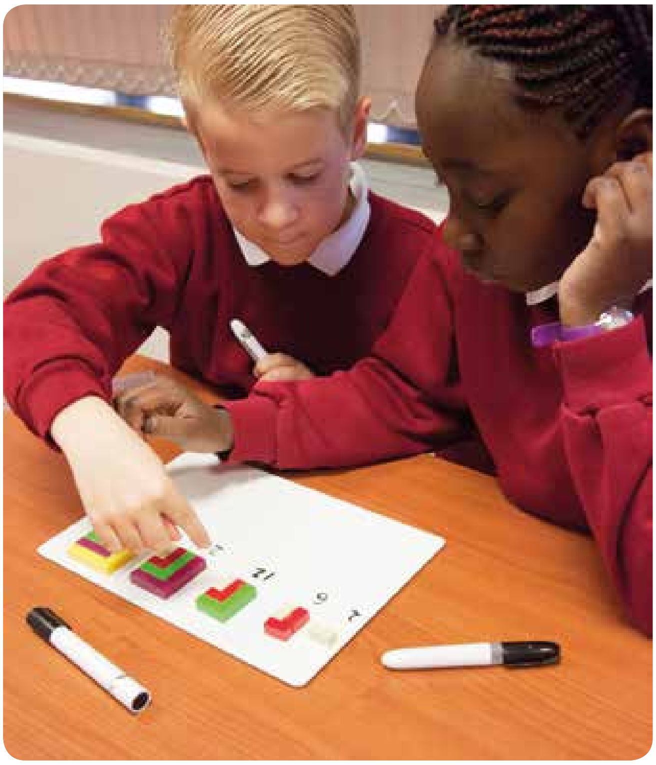
$$2x^2 - 7x + 3 = (2x - 1)\,(x - 3) = 0$$

and then deduce that x must either be $\frac{1}{2}$ or 3 because if:

$$(2x - 1)\,(x - 3) = 0$$

then *either* $(2x - 1) = 0$ or $(x - 3) = 0$ or both of them do, because the only way in which the product of two numbers can be zero is if either one, or both, of those numbers is itself zero. Hence, if $(2x - 1) = 0$, then 'x' must be $\frac{1}{2}$, or if $(x - 3) = 0$, then 'x' must be 3.

To carry out such algebraic thinking, children will need to know how numbers work with each other *in general*. This might be called thinking 'about' number relationships as opposed to thinking 'with' particular numbers (because in this case we don't actually know what all the numbers are). And so a large part of learning to do algebraic manipulation successfully involves coming to know how to manipulate numbers, whatever they are, and in particular when we *don't know* what they are. Non-computational thinking is thus an important introduction to key work on number relationships for children, as foundations are laid for their later mathematics. The work you do with children in the Number, Pattern and Calculating 5 activities on these aspects of calculating are already preparing them for success in their mathematics at secondary school.



Useful work on such non-computational thinking can also be developed as we invite children to think about changing a potentially difficult calculation into an equivalent but easier one, before they try to compute the specific answer. For example, 580 + 260 can be changed into (600 + 260) – 20, which is equivalent and (for many people) much easier to calculate mentally. Children at this stage should always be encouraged to think about any calculation they face *before they calculate*, rather than rushing in to calculate with the particular given numbers straight away. It is always possible that there is an equivalent calculation that would be easier to carry out.

Finally, tasks specifically requiring non-computational thinking can always be given to children simply to get them thinking in a 'relational' (algebraic) way. For instance, give children a statement like,

$$350 + 280 = 330 + 300$$

and ask them if they can explain why it is true *without* computing either addition calculation. The thinking required is non-computational (we don't want them to do the stated sums) and is also a great deal to do with seeing equivalence.

Functions – a special kind of pattern in mathematics

In Number, Pattern and Calculating 5, we continue to develop work on particular patterns that are called 'functions'. Mathematical functions are what we use to handle relationships of dependence between changing values, for example, the relationship between time, speed, and distance travelled as we move. Our distance travelled at any point depends on how fast we are going and how long we have been travelling.

Central to the general idea of a function is that of a 'variable'; this is what mathematicians call whatever it is that is changing. In our travelling example, the variables are time, speed and distance travelled, all of which change in relation to each other. Eventually, children will learn to handle functions symbolically with equations (or formulae). For example $s = \frac{d}{t}$, which could be read as, 'Speed is equivalent to distance travelled divided by time taken'. If you travelled 180 kilometres (d) in 3 hours (t), your average speed (s) was 180 ÷ 3, which equals 60 kilometres per hour. The quantities d, t and s are said to be 'variables' because they vary, but their interrelationship – the 'function', or how they depend on each other – remains the same. Functions are, importantly, another example of a generalization – or seeing the pattern of a relationship.

As an early example in the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, Pattern and Algebra 5, children were invited to generalize about the relationship between the value of a term in a sequence and its position in the sequence. For example, in the sequence 2, 4, 6, 8, 10, 12, 14 … the value, v, of any particular term depends on its position, p, in the sequence. We could write $v = 2p$ because the first term is 2, the second term is 4, the third term is 6, and so on. Classically, p is called the 'independent variable' and v is called the 'dependent variable', because as soon as you choose any particular position in the sequence, p, you know the value, v, precisely; put another way, the value of any term, v, *depends on* where it is in the sequence, p; that is, v depends upon p.

Sometimes the independent variable is called the 'input' to the function, and the dependent variable is called the 'output'. It is very important to remember that, for a relationship to be called a 'function' in mathematics, any given input must determine exactly one unique output (otherwise we wouldn't always know how v depends upon p, or indeed if it always does).

Generalizing this relationship between the value of a term and its position in the sequence allows us to *predict* an infinite number of values. We know, for instance, that the value of the 267th term will be 534, and the value of the 2 000 024th term will be 4 000 048 without having to write any of the previous terms in between to find out. This is an example of how, in so many ways, the generalizing we do in mathematics allows us to make relationships within our world predictable.

In the *Teaching Resource Handbook Number, Pattern and Calculating 4*, Pattern and Algebra 7, children were invited to use a letter to represent a variable, as they tried to generalize about the outcomes of multiplying odd numbers with even numbers and so on. In this case, they were invited to use the letter 'O' to represent any odd number and the letter 'E' to represent any even number – hence the values of 'O' and of 'E' vary; children were generalizing about the products of of *any* even and odd numbers.

In the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 1, children begin by revising 'linear sequences'. A number 'sequence' in mathematics is simply a collection of numbers set out in order of some kind; 2, 4, 6, 8, 10 … is a sequence of numbers. 'Linear sequences' are so-called because if we plotted a graph with them, that graph would produce a perfectly straight line; this happens only when the numbers involved have a constant difference (or step, up or down) between them.

The sequence discussed above, 2, 4, 6, 8, 10 … is a linear sequence, and if we were to plot the pairs of positions (p) and associated values (v) on a graph we would plot (p, v) to give the points (1, 2), (2, 4), (3, 6), (4, 8) and so on, which all lie along a straight line. Sensibly enough, the function relationship that connects values of v and p together ($v = 2p$) is also called a 'linear function'.

Children then go on to consider number sequences with more complicated rules, but at this stage, in Number, Pattern and Calculating 5, the crucial emphasis is upon showing these values *visually* (most often with number rods) so that children can literally 'see' how sequences are growing, and describe how terms grow using visual language. By then assigning number values to the rods, the growth of these sequences can finally be described in numerical terms. This work is laying a foundation for children later working out what the 'general term' of a sequence would look like, which is a preparation for the introduction of functions more generally.

Names for numbers: counting numbers and place value

Our civilization has been very clever in devising a system for generating symbolic number names, which not only allows us to go on inventing new names for counting numbers 'forever', but which also allows us to tell instantly where in the series of number names any particular name will be found. When we read '273' successfully, we know that it is the name of the whole number that comes immediately after 272 and a hundred before 373. This means that we don't have to remember every individual symbolic number name and its place in the order (which would be impossible anyway, since there are an infinite number of them); we just have to master the system that generates the names.

The two essential keys to generating this infinite set of names for counting numbers are that we 'group into tens', and that we use a writing code we call 'place value'.

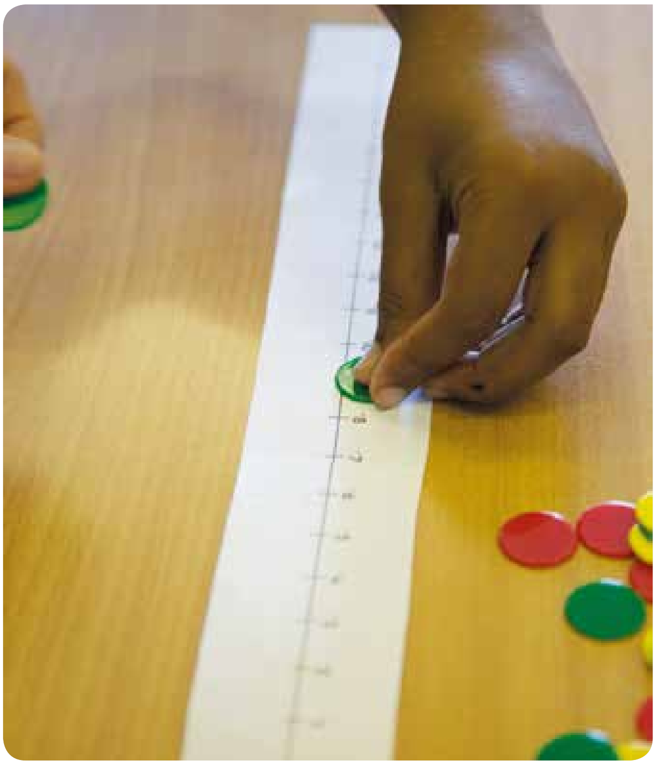
The first of these keys is **grouping** into tens. The number we call 'ten' (in numerals, '10') is the most important number in our naming system, because, when we are counting collections, as soon as we have ten of something, we call them 'one' of something else. So ten 'ones' are called one 'ten', ten 'tens' are called one 'hundred', ten 'hundreds' are called one 'thousand', and so on. In effect, in the language we use, we are always grouping things into tens (and then grouping groups) to call them one of something else.

In children's early experiences of finding how many objects there are in a collection, it was always important to help them to group collections physically into tens as they worked to find out how many objects they had. Finding 'how many?' by grouping in tens (and then possibly tens of tens) reminds children that our way of naming numbers uses a ten-based system. This idea remains crucial to their understanding of many calculating techniques in Number, Pattern and Calculating 5.

The second key to our symbolic number-naming system is **place value**, which is a kind of shorthand describing how the place of each digit within a string of digits signifies an important value. So, it is the place of '2' in the string '427' that tells us it has a value of 2 tens, or 20. It is important to realize, then, that the term 'place value' actually refers to a symbolic code for naming and reading number names, and that children have to learn either to crack the code or to reinvent it for themselves (depending on how they are taught).

Some people usefully distinguish between what is called the 'column value' and the 'quantity value' of a digit. For example, the column value of '2' in '427' is '2 tens', because it is in the 'tens' column, while its quantity value is '20', because that is its value as a quantity. In Numicon activities, we feel the important thing is that children understand that column value and quantity value are *equivalent*, that is, that the '2' in '427' means both '2 tens' and '20'; the two values are interchangeable. Children learn this equivalence through joining in our conversations around place value – another instance of the vital importance of our conversations with children when teaching mathematics.

The fact that children have spent time grouping objects in tens to 'find how many' in their early stages helps



children subsequently to partition numbers as part of many calculating techniques. For example, seeing 236 as '200 and 30 and 6', focusing upon quantity value, can sometimes (though not always) be the most helpful way of seeing the number, and in essence this partitioning is 'undoing' the grouping that they managed earlier.

Place value in Number, Pattern and Calculating 5

Children continue to develop their understanding of place value in Number, Pattern and Calculating 5 in three ways: by using larger and larger numbers, by developing their calculating methods, and by extending their work with decimal fractions. All of these aspects of number work depend crucially upon generalizing from a fundamental understanding of early *grouping* activities, and of our *place-value* code for number notation.

As in Number, Pattern and Calculating 4, process terms for grouping and re-grouping in tens, hundreds and so on are used explicitly, such as 'carrying', 'exchanging', 're-distributing' and 're-grouping'. Partitioning and re-combining numbers in these ways, as children calculate, are another instance of 'doing and undoing' actions – a key element of mathematical thinking for children to develop in many aspects of calculating (discussed further in the section on **inverse relationships**, on page 51). Children's continued practice in finding numbers on a number line in Number, Pattern and Calculating 5 also fosters an increasing confidence in ordering numbers and positioning numbers between other numbers.

As work with decimal fractions to three places develops, children will come increasingly to generalize the constant ratio relationships between 'places' (or columns) in our system of number notation. That is, children will increasingly use the regular '× 10' and '÷ 10' relationships between column values as we move to the left and to the right respectively in reading and writing multi-digit numbers. The significance of the decimal point marking out 'fractions' to the right of it is reinforced.

Negative numbers

Negative numbers are a very old idea, explicitly discussed and used in Hindu writings of the seventh century CE, and in much earlier Chinese calculating, where the colours of 'coloured rod' images denoted either a positive or negative aspect. In both cases, the context was accountancy, with negative numbers signifying an amount of debt. Calculating with these numbers was always dependent on the agreed sense they made in the practical context of money, loans, assets and debts.

Interestingly, for centuries, many European mathematicians resisted the idea that negative numbers are as valid mathematically as 'natural' numbers, refusing to allow that negative solutions to equations could be at all meaningful. Children having difficulties with negative numbers today have history on their side.

In Number, Pattern and Calculating 4, following the early Chinese and Hindu examples, children were introduced to negative numbers in practical contexts in which they make sense – temperature and underground car-parking levels. The important thing, in both contexts, is that in a real way amounts 'below zero' make intuitive and practical sense. More generally, the idea of negative versus positive amounts can often work in situations where there is some key point of reference that has meaningful amounts *on either side* of it, for example years before or after a key event on a timeline, or travelling towards or away from a geographical 'zero' position on a physical line.

The essential thing for children to understand when they meet negative numbers is that from now on numbers may be considered to have not just a 'size' but also a 'direction'; this is why integers (positive and negative whole numbers including **zero**) are sometimes together called 'directed numbers'.

Importantly, using a 'zero' point on a physical line is what prepares children for the imagery of negative numbers presented as distances to the left of zero on a conventional number line. In Number, Pattern and Calculating 5, children continue to meet negative numbers in context, count forwards and backwards along a number line using both positive and negative numbers, and begin to calculate by finding differences between directed numbers.

In Number, Pattern and Calculating 5, children are also introduced to the important idea of *ordering* directed numbers, and this raises the intriguing question of whether $^{-}12$ is 'bigger' or 'smaller' than $^{-}7$. At this stage, it is always best to answer in context; so we could say $^{-}7°$ C is warmer than $^{-}12°$ C, and that someone who owes £7 is better off than someone who owes

£12. Otherwise it is probably best to focus simply on direction when ordering directed numbers and ask, e.g.: 'Which number is *to the right of* the other one on a number line?'

Much later on in their school life, and in other practical contexts, children will meet the mathematical term 'vector'. A vector is something with both magnitude (size) and direction. Usually, vectors make initial sense in the context of physical forces – which have both a strength and a direction – and once children get used to the idea of vectors they can often look back to meeting negative numbers and realize that, once a number line includes both negative and positive numbers, *every* number on it will have both magnitude and direction and is therefore also a 'vector'.

Factors and multiples, prime and composite numbers, squares and cubes

From the *Number, Pattern and Calculating 3 Teaching Resource Handbook* onwards, as multiplication tables become more committed to memory, and as children experience more and more calculating, their familiarity with numbers increases and it becomes possible to introduce new categories of numbers. In the Number, Pattern and Calculating 5 activities, children learn to talk in more detail about different kinds of numbers and number relationships as they multiply and divide, and the following terms become important: factor, multiple, prime, composite number, square number and cube number.

As children learn their multiplication tables and thereby gradually become more familiar with multiplying and dividing relationships between numbers, they should be encouraged to make – and to explain – observations such as '36 seems to crop up in lots of places'. This happens because 36 is the product of 3 × 12, 12 × 3, 4 × 9, 9 × 4 and 6 × 6; we can use observations like this to introduce children to the term '**factor**', meaning a number that divides into another number exactly, without leaving a remainder. Numbers 3, 4, 6, 9 and 12 are all called factors of 36 because they divide exactly into 36 without leaving a remainder; 1 and 36, and 2 and 18 are also factors of 36. In this way, we can encourage children to explain their observation as, '36 seems to crop up so many times because it has a lot of factors'.

Afterwards, we can also put the relationship the other way around and say that since 1, 2, 3, 4, 6, 9, 12, 18 and 36 are all factors of 36, that means, conversely, that 36 is a '**multiple**' of 1, 2, 3, 4, 6, 9, 12, 18 and of 36.

Not all numbers have as many different factors as 36 does; some numbers have only two different factors, and such numbers are called '**prime numbers**'. 3, 5, 7, 11, 13, and 17 are all prime numbers; each of these is divisible only by 1 and by itself; 3, for example, is divisible only by 1 and 3.

Any positive whole number that is not a prime number is said to be a '**composite number**'. 36 is a composite number because it is a positive whole number, and it has many more than two different factors.

When a number is multiplied by itself, the product is said to be a '**square number**', probably because the area of a square is calculated by multiplying the length of its side by itself. 36 is a square number since it is the product of 6 × 6.

When a number is multiplied by itself twice, the product is said to be a '**cubic number**' (or a 'cube'), probably because the volume of a cube is calculated by multiplying the length of its side by itself, twice. 8 is a cubic number since it is the product of 2 × 2 × 2.

In Number Pattern and Calculating 5 children also begin to be introduced to tests of what is called 'divisibility'; these are ways of testing whether a number has a particular factor, or not. For example, a number has a factor of 4 if the last two digits are also divisible by 4, e.g. 6724 is divisible by 4, as 24 is divisible by 4. This is because, since 100 is divisible by 4 then any number of whole hundreds will be divisible by 4, so as long as any number's last two digits are divisible by 4, it will have a factor of 4.

Fractions, decimals, ratios and percentages

Fractions involve a complex set of relationships and, confusingly for many children, there are several different symbolic ways of representing what are essentially the same numbers, e.g. $\frac{3}{5} = \frac{9}{15} = 0{\cdot}6 = 60\% = 3{:}5 = 3 \div 5$. One of the key challenges for teachers at this stage is to guide children to understanding that common fractions, decimal fractions, percentages, ratios and dividing calculations, are essentially different forms of notation for expressing the same 'rational' numbers, and that 'ratio' is at the heart of multiplicative thinking (see the 'Multiplicative thinking' section on page 65).

Typically for children, fractions *of* things arise in measuring situations, which importantly include 'sharing'. The measuring of continuous quantities, such as time, length or chocolate and so on is always approximate and for this reason we commonly find ourselves needing parts of whole units to describe amounts accurately. The moral imperative for fair shares usually draws children easily to the view that fractions are, and indeed should be, about *equal* parts (or proportions) of a whole.

The two main ways in which children experience fractions initially in Numicon activities are therefore as 'operators' and as 'descriptors'– fraction words used as verbs and as adjectives. An initial invitation to '*halve* twenty-six' would be an invitation actively to *find* 'half' of 26 – the fraction word is used as part of an instruction to *do* something. Then, to describe the outcome of some measuring tasks, or of some dividing calculations, children would use fraction words as adjectives, for example in the description 'twenty-six-and-a-half *somethings'*, or as the description of a relative distance, for example as 'halfway' between 26 and 27 on a measuring scale.

It is important to note too that in work from the *Number, Pattern and Calculating 2 Teaching Resource Handbook* onwards, children were also meeting fractions as *objects* (as numbers in themselves), signalled by the use of fraction words as *nouns* and by their representation along a pure number line. From introducing halving situations in which there was an action of halving (say) a pizza, and in which children used the verb, 'to halve', we moved to describing actual amounts of things using fraction words as adjectives; then we subsequently asked them to do a very strange thing, which was to start talking about 'a half' as a rather isolated abstract object, using the word 'half' on its own, as a noun; and in which mathematical context, any trace of pizza (or of anything else from a material world) has disappeared altogether.

It cannot be emphasized too strongly that such talk of actions (using verbs), turning to talk of 'fractions *of* something' (using adjectives), turning to talk of just isolated fractions as mathematical objects (using nouns) involves very significant changes in our communicating for children to join in with. The models and imagery we offer children to help them to get used to these strange new objects in our communicating are crucial, and once again children need plenty of time, opportunity and imagery to get fully used to our new ways of talking and communicating about them.

Interestingly, the shift from fraction words used as adjectives to their use as nouns mirrors something that happens much earlier with our communicating about whole numbers. When children are first introduced to whole numbers as objects, and numerals begin to be used on their own, e.g. as in $3 + 6 = 9$, it can be helpful to suggest to children that they read the number sentence as, '3 *of anything* + 6 *of anything* = 9 *of those things*'. This makes it explicit that '3' used on its own, as a noun, is a *generalization* that means '3 *of anything*'.

In the same way, when using fraction words as nouns and beginning to ask children questions such as $\frac{2}{5} + \frac{4}{5} = ?$, it can be helpful to suggest that children read the number sentence as, '$\frac{2}{5}$ *of anything* + $\frac{4}{5}$ *of that thing* = what?' This may help children to understand that fraction words and symbols used on their own, as nouns, are also *generalizations*, and that '$\frac{2}{5}$' used on its own means '$\frac{2}{5}$ *of anything*'.

In the *Number, Pattern and Calculating 1* and *2 Teaching Resource Handbooks*, common (or 'vulgar') fractions and their conventional notation begin to be introduced, and as new number objects they began to be related to existing whole numbers through also representing them as distances along a number line. In the *Number, Pattern and Calculating 3 Teaching Resource Handbook*, the terms 'numerator' and 'denominator' are formally introduced, counting on and back in fractions along a number line is further developed, and some fractions (< 1) with the same denominator are added and subtracted.

In the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, key developments involve the introduction of decimal fractions, mixed numbers and improper fractions, and, importantly, recognizing the *equivalence* of a range of common fractions (< 1), for example $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$, which will prove to be fundamental to later methods for adding and subtracting fractions with different denominators.

In the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, work continues with recognizing equivalence between common fractions, and also now between improper fractions and mixed numbers, between common fractions and decimal fractions, and between common fractions, decimal fractions and percentages.

Most importantly, strong explicit connections are made between common fraction notation and the process of dividing, i.e. recognizing that $\frac{5}{8}$ is equivalent to 5 ÷ 8, and it can take some time for children to accept an equivalence between an action (dividing) and an object (a fraction). Allow for plenty of discussion about this.

Children are also introduced, in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Numbers and the Number System 6, to the challenges of *ordering* fractions with different denominators, an activity that normally requires first converting given fractions into *equivalent* fractions with a common denominator. Equivalence is important everywhere.

Work on adding and subtracting fractions also develops further in Number, Pattern and Calculating 5 (see the sections on **adding and subtracting**, below), and multiplying fractions by whole numbers is introduced (see the section on **multiplying and dividing**, page 61). The adding and subtracting of fractions in the Number, Pattern and Calculating 5 activities is both extended and restricted to the adding and subtracting of fractions whose denominators are multiples (or factors) of each other, e.g. $\frac{1}{3} + \frac{5}{6}$.

Finally, in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Numbers and the Number System 7, children are introduced to that important set of fractions called 'percentages', or fractions with a denominator of 100. The importance of percentages is that they are used primarily in contexts where quick or ready *comparisons* are important, and this probably explains their origin in the context of trade – in which a rapid comparison of offers is often crucial. Proportions are often expressed in percentages as well, since the notion of a 'whole' (something) that is necessary to understanding proportion is readily expressed as '100%'.

We discuss the important relations between fractions and ratios (which are also introduced in Number, Pattern and Calculating 5) in the section on **multiplicative thinking**, page 65.

The use of Numicon Shapes, number rods and objects arranged in arrays, and visual imagery such as diagrams and number lines, continues to be essential to communicating about fractions, as is the use of everyday and realistic contexts that children can relate to. Measuring scales are particularly useful. When using the terms 'numerator' and 'denominator' with older children it can also be helpful to explain their sense. A de*nom*inator gives a common fraction its name – it tells you what kind of a fraction it is. A *num*erator tells you how many of this kind of fraction you have. There is always a history to how we do and say things in mathematics.

Arithmetic operations, or 'the four rules': adding and subtracting

In Number, Pattern and Calculating 5, children continue to develop their adding and subtracting, with an emphasis now more upon written methods necessary for calculations too difficult to be accomplished purely mentally. All calculating involves working mentally, but some calculating with small whole numbers, or with very convenient larger numbers, can be purely mental. Children should always be encouraged to think about a calculation first, before simply diving in thoughtlessly with the first method that occurs to them. Non-computational thinking may often reduce an apparently difficult calculation to a much easier one that can be carried out purely mentally (see the section on **non-computational thinking**, page 53).

In Number, Pattern and Calculating 5, there is increasing illustration of written 'column' methods of adding and subtracting with base-ten apparatus. Teachers need to be very clear that we do not want children to think that they have to actually 'do their sums' with this apparatus. The use of these materials is purely to *illustrate* number equivalence relationships in ways that reflect our place value system of naming numbers; actions with the materials are used simply to help children to think and communicate about how the 'grouping' or 'exchanging' involved in most written methods *makes sense*, not as a method of calculating in itself.

Thus both teachers and children need to be very clear that using base-ten apparatus to *illustrate* how numbers are manipulated during a calculation is not in itself a method of calculating; actions with materials are not carried out to *produce* an answer, but to *explain* the number actions involved in calculating.

In Number, Pattern and Calculating 5, children also begin to add and subtract both common fractions and decimal fractions, and these operations will test children's understanding of both equivalence and place value respectively.

With all kinds of numbers however, children continue to work on the following aspects of adding and subtracting in the Number, Pattern and Calculating 5 activities:

- **structures** – the different kinds of situation in which adding and subtracting occur; and
- **methods** – how to calculate.

Structures for adding and subtracting

Within the Numicon activities, we address two adding structures – **aggregation** and **augmentation** – and children should be given regular experiences with both.

Aggregation is putting together. Two or more amounts or numbers are put together to make a 'total' or 'sum'. For example: 'I had £20. John gave me £10, and Nana gave me another £35. How much did I have in total?'

Augmentation is about increase. One amount is increased or made bigger. For example: 'Special offer! One third extra, free!'

We expect children to recognize four subtracting structures:

take away, **decrease**, **comparison** and **inverse of adding**.

Subtracting is more complex than adding as it is more varied in the different kinds of situations in which it occurs. Again, children should be given experience with all four structures regularly.

Take away refers to those situations where something is lost, or one thing is taken away from another. For example: 'Gemma had £19 at the beginning of the day. She spent £6·47. How much does she have now?'

Decrease is about reduction. For example: 'Special offer! 25% off!'

Comparison is where two amounts are being compared and we want to find the additive difference. For example: 'Samir has saved £34·40 for his holiday, and Nihal has saved £42·65. What is the difference between the amounts of money that Samir and Nihal have?'

As comparisons involving negative numbers are introduced, comparisons and differences between ranges of positive and negative numbers can be helpfully illustrated using a continuous number line.

The **inverse of adding** structure is about wanting to know how much more of something we want or need in order to reach a particular target. For example: 'The blue trainers cost £59·50. I have £38·25. How much more do I need to buy the shoes?' Children can often feel very confused about adding on in order to accomplish subtracting, and it is important for teachers to be clear about what is going on here. The reason this adding manoeuvre is included as a subtracting structure is because the adding on in these cases is done in order to find out a *difference*; in most adding we know how much to add, and we do it.

Methods for adding and subtracting in Number, Pattern and Calculating 5

There is more emphasis in Number, Pattern and Calculating 5 on developing written methods of adding and subtracting with larger numbers, but children should always be encouraged to think before they act. In particular, children should always be encouraged to think first about whether any given calculation could be transformed into an easier, equivalent calculation, and secondly to estimate what the answer is likely to be – approximately.

Non-computational thinking should thus become a habit as children are increasingly asked to think about their calculating, rather than just responding mechanically to an addition or subtraction sign. Is there the possibility of altering a calculation to an equivalent 'easier' one – for instance, altering 840 – 380 to (840 – 400) + 20? (Adjustments of this particular kind are sometimes called 'rounding and compensating'.) Children should regularly be encouraged to notice how useful the basic number facts to ten are in calculating with larger numbers, for example in being able to generalize from 6 + 4 = 10 to 60 + 40 = 100, and to 600 + 400 = 1000.

Non-computational thinking and estimating answers to calculations in advance should become a habit for children when using *any* kind of numbers, but especially

when adding and subtracting fractions and decimals. This will help to develop their understanding of these numbers enormously.

When adding and subtracting fractions, children will find that just as with currencies of different denominations, it is not possible to add (or subtract) fractions of different denominations. In the same way that we cannot add or subtract \$6 and £3 together directly, so we cannot add or subtract $\frac{4}{5}$ and $\frac{3}{8}$ together directly; in both cases we have to transform the amounts into *equivalent* amounts in a *common* denomination. We either have to convert \$6 to £ (or £3 to \$), or convert both to a common third currency (e.g. €) before we can add or subtract them; with fractions, we would normally convert each of the two above to equivalent numbers of fortieths (their lowest common denominator).

When adding and subtracting numbers involving decimal fractions, children's existing understanding of the place-value notation they use for naming whole numbers will be the basis from which they *generalize* to bring meaning to the columns to the right of the decimal point. In particular, children will need to be clear that the 'value' of a column 'place' divides by ten each time we move one 'place' to the right. Thus the columns to the right of the decimal point have values of one tenth, one hundredth, one thousandth, and so on as they move to the right.

Finally, in the Number, Pattern and Calculating 5 activities, children begin to generalize the technique of '**bridging**' through multiples of ten to bridging through larger multiples (e.g. 100), and also bridging with fractions through different kinds of convenient 'whole' points in different contexts. For example, when adding $\frac{4}{5}$ and $\frac{3}{5}$ together, we can mentally partition the $\frac{3}{5}$ into ($\frac{1}{5}$ and $\frac{2}{5}$), add the $\frac{1}{5}$ on to the $\frac{4}{5}$ first, and thus 'bridge' through 1 to obtain the answer '$1\frac{2}{5}$'. Note that we can also 'bridge through 1' if we were doing the same calculation with decimal fractions: 0·8 + 0·6 can be managed as 0·8 + (0·2 + 0·4), which is the same as (0·8 + 0·2) + 0·4, which then gives the total '1·4'. Bridging thus becomes an invaluable mental technique for adding and subtracting in a wide range of contexts, and particularly when using the units of various measures as bridging points; when adding 350 m to 1 km 800 m for example, it is helpful to 'bridge' through 2 km.

Arithmetic operations, or 'the four rules': multiplying and dividing

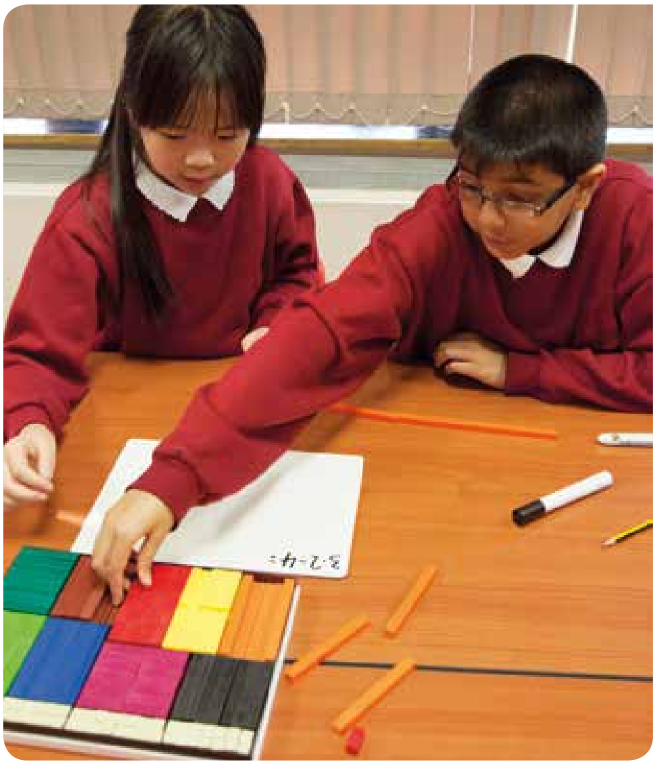
In Number, Pattern and Calculating 5, work continues on straightforward multiplying and dividing situations and calculations with larger whole numbers, using both purely mental methods and written methods. As with adding and subtracting, children should be encouraged always to think before they act; non-computational thinking will often reveal ways of making an apparently difficult multiplying or dividing calculation much simpler, and children should also get into the habit of always estimating what an answer is likely to be, approximately.

The calculation 36 × 25 = ☐ for example, can be re-interpreted as (9 × 4) × 25 = ☐ using a particular pair of factors of 36, and then as 9 × (4 × 25) = ☐ (using the associative property), which is then easy to calculate purely mentally. In this case the 'non-computational' recognition and use of factors removes the need for any laborious form of 'long' written multiplication.

Also, as with children's work on adding and subtracting, base-ten apparatus is often used in the Number, Pattern and Calculating 5 activities to illustrate the number relationships involved in 'partitioning', 'exchanging', 'grouping' and so on that feature within some written methods of multiplying and dividing with larger numbers. It is again important to establish that base-ten apparatus – as with all other materials and imagery used in our approach – are used to *illustrate relationships*. Actions with physical materials are not used as 'methods' for producing answers, but to *explain* the sense of number actions that are used in calculating with numerals.

Most significantly, in Number, Pattern and Calculating 5, children begin to multiply with both common fractions and with decimal fractions, and this begins to deepen their encounters with multiplicative thinking considerably. 'Multiplicative thinking' is a term increasingly used now to refer to a whole set of different ways of thinking about *comparison relationships*, and is usually contrasted with 'additive thinking'.

Multiplicative thinking is essentially about comparisons involving ratios, whereas additive thinking is essentially about comparisons involving additive differences; for example, 6



is 'four more than' 2 (an additive comparison), whereas 6 is 'three times as big as' 2 (a multiplicative comparison).

Multiplying and dividing

The operations of multiplying and dividing have several aspects, not all of which make immediate sense to children. In Number, Pattern and Calculating 5, we continue to distinguish between the **repeated adding**, **ratio** (or **scaling**) and **array** structures of multiplying. Repeated adding and scaling up are often fairly intuitively understandable and have been built on children's earlier experiences of counting on in 2s, 5s and 10s, and doubling, since work in the *Number, Pattern and Calculating 2 Teaching Resource Handbook*.

As with our discussion of adding and subtracting, in relation to multiplying and dividing we address:

- **structures** – the different kinds of situation in which multiplying and dividing occur; and
- **methods** – how to calculate.

Structures for multiplying

Repeated adding is the familiar 'so many lots of something' idea, in which equal amounts are added. For example, '12 tables each need 8 place settings. How many place settings are needed altogether?'

Ratio is the 'multiplying up' idea we use when we want to scale something up, for example, making a recipe for 6 people instead of 2.

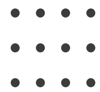
Both of these structures have very important and strong inverse connections with dividing. Scaling up, for instance, is associated with its inverse in dividing, of scaling down – for example, halving.

Importantly, children should be encouraged to notice that when two numbers are being multiplied together to give a product, in many situations each number plays a different role – one number refers to an amount being multiplied (technically, this is the 'multiplicand') and the other determines how many times that number is to be multiplied (the 'multiplier'). One number is being multiplied; the other does the multiplying. In practice, we teach children pretty quickly that multiplying has a commutative property so, for example, 4 × 6 = 6 × 4 and that in a sense it doesn't matter which number is doing the multiplying, as the product will be the same.

However, this practice of saying 'it doesn't matter which one is the multiplier' can turn out to be unhelpful to children when they later try to make sense of the inverse connections between multiplying and dividing. Seeing dividing as the inverse of multiplying (and using our multiplication tables to solve dividing problems) is a bit like turning a dividing problem around and saying, 'We already know the product of two numbers, but we only know one of the numbers that were multiplied together.'

And in practical situations, it can make a difference whether the number we know is the multiplicand or the multiplier. If we know the multiplicand (the size of the groups), we then want to know how many times that number goes into the product; if we know the multiplier, or how many times something goes into the product, we want to know how big that 'something' (the multiplicand, or size of the group) was. The first case applies to dividing situations, such as working out how many 15-seater minibuses we need to ferry 60 children around (this is called 'quotition'). The second applies to sharing situations, such as, 'How much will we each get of that cake?' (this is called 'partition').

The third multiplying structure, that of an **array**, will help children to see the **commutative property** of multiplying and it will also help them to connect multiplying with the measurement of area, to understand how multiplying by fractions makes answers smaller, and to understand how the multiplication of large numbers can be broken down into smaller calculations (the **distributive property**. All of this involves interpreting multiplying as an array, for example illustrating 3 × 4 as:



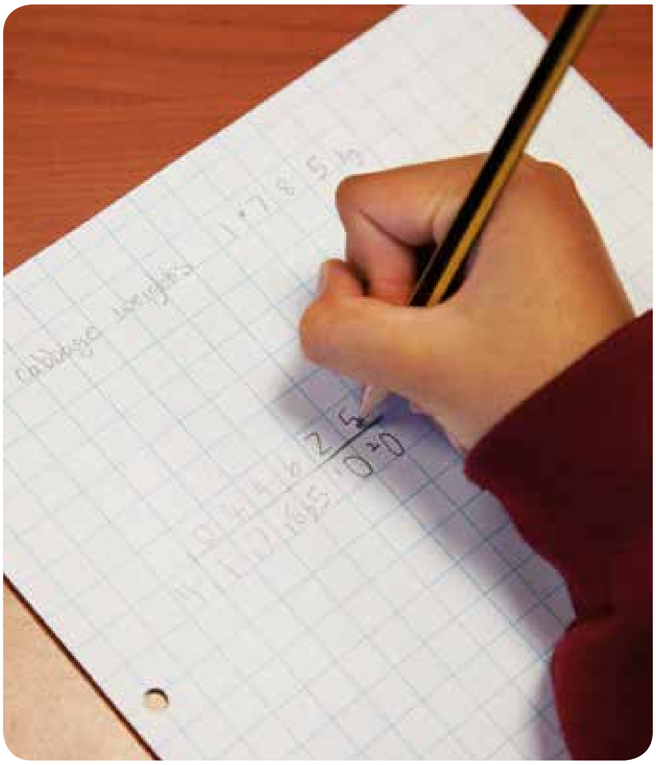
In the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, we placed increased emphasis on arrays as children worked to generalize the distributive property – that is, that multiplication is 'distributive over addition'. It is this property that underlies the 'grid method' of multiplication that was also introduced at that stage (see the section on the **distributive property**, page 52). Arrays are also helpful to children when facing **correspondence problems** such as, 'I have 5 T-shirts and 3 hats; how many different outfits can I put together?' (see the section on **multiplicative thinking**, page 65).

The Number, Pattern and Calculating 4 activities, also involved illustrating multiplying as repeated adding using number rods laid end to end, and then rearranging the rods alongside each other to form arrays, thus visually uniting the repeated adding and array structures. As all Numicon Shape patterns are themselves simple arrays of holes, children following Numicon activities will have already been familiar with the idea of arrays informally from their earliest experiences with the materials.

The **ratio** structure of multiplying was developed in the *Number, Pattern and Calculating 3* and *4 Teaching Resource Handbooks* in terms of everyday situations that required some scaling up, for example, of recipes. In order to establish essential links with dividing from the beginning, the images and patterns developed in the activities for multiplying are usually quickly exploited to illustrate dividing; recipes are also scaled down to offer a context for the **ratio** structure of dividing.

Note on a reading convention: When reading and recording multiplying sentences such as 4 × 7 = 28, there are many choices of interpretation and often a surprising amount of controversy about whether '4 × 7' *really* means 'four 7s' or 'seven 4s'. Of course, the array structure quickly demonstrates that their product is the same, but some teachers feel that only one reading of the sentence can be 'mathematically correct'.

The truth is that we do have choices, and that there are equally good reasons for choosing either way. In Numicon activities, we have chosen to introduce reading '4 × 7' as 'four times seven', meaning four lots of seven, for a number of reasons: in order to exploit the everyday use of the word 'times' (signalling repeated actions); to tie in with the traditional way of reading and saying multiplication tables in the UK; to be consistent with conventions for units of



measure – for example with the meaning of 3 kg as three 'lots of' a kilogram; and to be consistent with algebraic expressions such as $4x + 3y$ (commonly interpreted as '4 lots of x' and '3 lots of y').

Structures for dividing

There are three essential structures of dividing: the **grouping**, **sharing** and **ratio** (or **scaling down**) structures.

The **grouping** structure – technically called 'quotition' – occurs in situations where we know an amount, the 'dividend', and we want to know how many times a different amount, the 'divisor', will go into it. This type of situation will lead to remainders when the divisor is not a factor of the dividend. For example, 8 goes into 43 five times, leaving a remainder of 3. We often call this '8 divided into 43' (or '8s into 43').

The **sharing** structure – technically called 'partition' – occurs in situations where, again, we know the amount to be shared, the dividend, and this time we know how many equal parts the dividend is to be shared into but we don't know how big each share will be. This type of situation will lead to fractions when the number of shares, the divisor, is not a factor of the dividend and the object(s) being shared can be broken into parts. For example, 3 chocolate bars shared between 2 people will give each person $1\frac{1}{2}$ bars. We might call this '3 divided into 2 parts' (hence sharing is called partition). Note: this type of situation may also help children to understand the equivalence of $\frac{3}{2}$ and $3 \div 2$.

It is important that children learn to distinguish between these two types of situation if their answers to dividing problems are to make sense. If we have a situation where we have some money, for example £32 (the dividend), and we want to know how many tickets each costing £1·50 we can buy with the money, the answer is 21 remainder 50p, not (as a calculator would show) 21·3333 … . This is a grouping (or quotition) problem; we want to know how many times £1·50 goes into £32 – there is no sharing involved and fractions make no sense as an answer. This is £1·50 into £32.

On the other hand, if 3 children are going to share 10 fish fingers (the dividend) between them (fairly!) their equal shares will be 3·3333 … ($3\frac{1}{3}$) fish fingers each. In this situation (partition), we already know how many parts the fish fingers are to be divided into (3), but we don't know how big the resulting equal parts will be. This is 10 shared into 3 equal parts.

Unfortunately for children (once again), the language we use when speaking of dividing calculations is often confusing. As you may already have noticed, we tend to use the word 'into' for both sorts of situation – 'dividing 3s into 10' (quotition), as well as 'sharing 10 into 3' (partition). We also often speak of 'dividing 10 by 3'. To complicate things further, when we introduce mathematical symbols for dividing to children, we often tend to imply different structures for the calculation rather carelessly. For instance, $10 \div 3$ tends to be read as 'ten divided by three' and is often explained as a 'sharing' problem, whereas $3\overline{)10}$ tends to be read as 'threes into ten', probably because (reading conventionally from left to right) the numbers in this second case appear in the reverse order and we want children to use their tables as they solve it. In both cases, we want the children to do the same dividing calculation, but these symbols are often explained as two quite different structures of dividing (the first partition, and the second quotition) and children can very reasonably struggle to understand exactly what it is we want them to do. Do we want them to share 10 into 3 (partition), or to find how many 3s in 10 (quotition)?

Eventually children will come to understand that dividing can be seen either way, but that the way we see a dividing problem will affect the kind of answer we give. Mixing up both structures of dividing without making them distinct to children often leads to confusion, and to answers that don't make sense. In particular, children often struggle to understand what to do with remainders (see page 64).

The ratio structure of dividing occurs when something is being scaled down, for example when a scale model – usually of something large – is made. The classic examples are of course maps, in which large actual geographical distances are all divided by the same number to produce an image that fits onto a piece of paper.

Numicon activities initially introduced dividing as grouping (quotition) and left sharing (partition) and fractions resulting from a dividing calculation for a different lesson. This was done in order to put some mental space between initial experiences of the two structures. Simple fractions were of course discussed with children, but usually in the context of incomplete units of measure, and not as ways of dividing up whole-number remainders of division calculations.

So, in Numicon activities, we have always introduced dividing firstly and distinctively as the grouping structure, which emphasizes its inverse relation to multiplying. Having shown how 7 × 3 = 21 in multiplying, we then connected this with dividing as grouping (as quotition) by asking, 'If 7 times 3 is 21, how many 3s are there in 21?' In Number, Pattern and Calculating 5, we continue to emphasize the inverse relationship between dividing and multiplying at every opportunity.

The ratio structure of dividing was introduced in Number, Pattern and Calculating 1, with halving, and subsequently children have also been invited to find thirds, quarters and sixths of various quantities. This can also be seen as multiplying by a half, a third, a quarter, and so on (see the section on **inverse relationships**, page 51), but children are not asked to make this connection with multiplicative inverses explicit yet. Contexts featuring the ratio structure of dividing continue to figure in Number, Pattern and Calculating 5 activities.

A special note about remainders

When a dividing calculation involving whole numbers doesn't work out exactly, we divide the dividend by the divisor so far as we can, and then find we have a small whole number 'left over'. Children are often confused about what to do with the leftover number; sometimes we leave it as a 'remainder', and sometimes we carry on dividing the remainder into fractions (or decimals). When should we do which? The answer always depends upon the context in which the dividing has arisen, and often the difficulty for children is that there are two quite different reasons for leaving a remainder and a third reason for going into fractions.

The first reason for leaving a remainder is that we are dealing with a quotition situation and fractions would not make sense. If 23 people need taxis home and a taxi will take five people, we divide 5s into 23. The solution to this problem is not that we need $4\frac{3}{5}$ taxis, but that hiring 4 taxis will leave a remainder of 3 people unable to get home (so we'd better order 5 taxis).

The second reason for leaving a remainder is totally different: in a partition situation, we might find ourselves sharing out objects that cannot be broken into smaller parts. If 3 children have 50p to share between them, they can have 16p each and there will be 2p left over that cannot physically be broken down into three equal parts – it has therefore to be left as a remainder.

The reason for continuing to divide a leftover whole number into fractions occurs in a partition situation in which the thing being shared can actually be broken down into smaller parts. (The word 'fraction' comes from the same etymological root as the word 'fracture'.) If 6 people are sharing 4 pizzas (4 ÷ 6 as a dividing problem), the solution is not 0 remainder 4 pizzas (everyone gets nothing), but that each person gets $\frac{2}{3}$ (or $\frac{4}{6}$) of a pizza. Note again the usefulness of a situation like this for illustrating the **equivalences** between 4 ÷ 6, $\frac{4}{6}$ and $\frac{2}{3}$.

Children need much experience with and illustrated discussion of all three kinds of situation.

Methods of multiplying and dividing

As with adding and subtracting, we work on developing children's 'fluency' in multiplying and dividing primarily through ensuring that all work is grounded in a depth of understanding of the natures of these operations, of the types of context they are relevant to, and of how (in our base-ten system) both numbers and operations 'fit together'.

As with adding and subtracting, when multiplying and dividing, children should get into the habit of always asking first whether non-computational thinking might allow them to change any calculation into an easier (or more convenient) form, and also to estimate what any product or quotient is likely to be. For example, 19 × 45 = ☐ can be thought of as (20 × 45) – 45 = ☐, and then as 900 – 45 = ☐, which we would by now expect children to do mentally. Such thinking and estimating is particularly important when dealing with fractions and decimals, to support children in building their understanding of multiplicative thinking.

In a scaling-down situation, such as modifying a recipe for fewer people, children might need to find a third of 250 g of flour, i.e. $\frac{1}{3}$ × 250 g. It will probably help them to be able to translate this multiplying calculation into the division 250 ÷ 3, to do the dividing, and then to reason that neither 83·333 … nor '83 and one left over' are helpful answers in this context, so that 250 g ÷ 3 ≈ 83 g (or even ≈ 80 g) will be the most sensible answer. This is an example of what is called multiplicative thinking in context, even though the calculating involved turned out to be dividing.

Children need to learn their multiplication tables, and in the Numicon activities these essential facts have been introduced in an organized sequence, exploiting available patterns along the way to ensure the tables are both connected with each other and memorable. Use of multiplication tables allows children to cope with multiplying small or easy numbers mentally and (as with adding and subtracting) mental methods should always be considered as a first resort. Recalling tables facts quickly is essential to fluency with all written methods of multiplying and dividing.

There is emphasis upon multiplying and dividing by increasingly large multiples of ten in the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, partly as the easiest examples of scaling up and down to calculate (in our base-ten system), and partly also to support children's extension of whole-number multiplying and dividing into multiplying and dividing with decimal fractions.

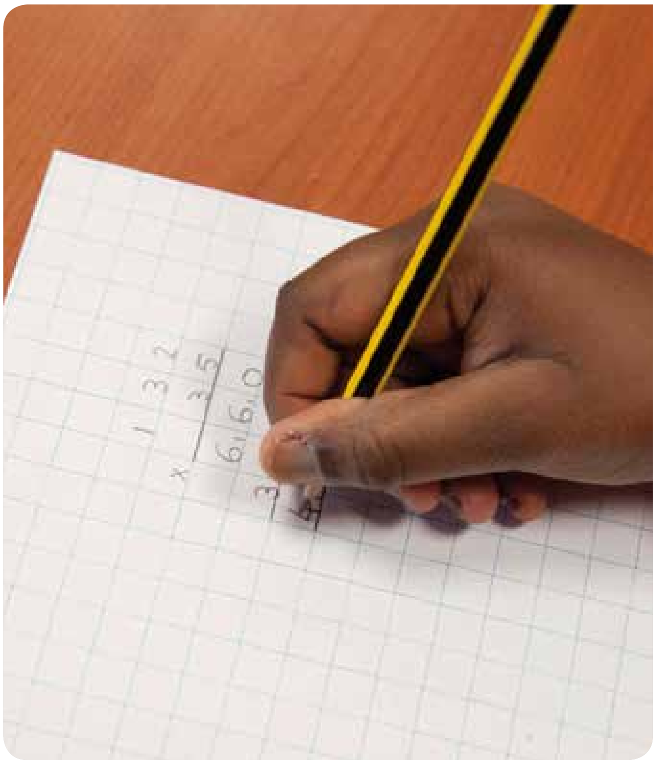
Written column methods of short multiplying and dividing were developed in the *Number, Pattern and Calculating 4 Teaching Resource Handbook*, and are extended in the Number, Pattern and Calculating 5 activities to include decimal fractions. The 'grid method' of multiplying – crucially built upon experience with visual arrays – is continued as well, but in Number, Pattern and Calculating 5, this is now complemented with the more traditional 'long multiplication' column method, both methods in fact reinforcing understanding of the distributive property, albeit in different ways.

In the Number, Pattern and Calculating 5 activities, children also begin multiplying common fractions, decimals, and mixed numbers by whole numbers. Such calculations are often important within measuring contexts involving derived measures, such as finding the paved area of a pathway 7 m × $\frac{1}{2}$ m. This work is also importantly connected with multiplying and dividing to find *fractions of* particular amounts, i.e. recognizing the equivalences between $\frac{5}{8}$ of 23, $\frac{5}{8}$ × 23 and (5 × 23) ÷ 8.

Multiplicative thinking

There is no doubt that what has come to be called by many people 'multiplicative thinking', or sometimes 'proportional reasoning', is a distinctive way of thinking about relationships that underlies much later work for children, both in mathematics and in science. It would be fair to say that without developing their multiplicative thinking, children will be unable to cope with large swathes of their secondary schooling, or indeed with many crucial aspects of their everyday and working lives. Both our physical universe and our social worlds are filled with multiplicative relationships that we need to be able to manage.

To name a few, sharing, cooking, making drinks, preparing medicines, regulating doses, maps, models, making predictions, assessing risks, measuring speeds, anticipating income, planning spending, engineering, doing science,



converting currencies, designing anything and comparing performances all use multiplicative thinking.

The root idea behind multiplicative thinking is that of a ratio comparison (or correspondence), and this is typically contrasted with additive comparisons. There is some evidence that children typically latch on to additive comparisons (for example, 'He's got more than me!' and 6 = 3 + 3) significantly before they learn to make multiplicative comparisons (for example, 'He's got twice as much as me!' and 6 = 2 × 3), and that ratios and proportions are almost universally much harder for us to think about than additive relationships are.

It is important to remember that although this type of thinking is called 'multiplicative', it often involves dividing. This is because multiplying and dividing are essentially 'two sides of the same coin'. As inverse operations, each one may 'undo' the other, and almost every dividing problem can be converted into an equivalent multiplying problem, for example 17 ÷ 6 = 17 × $\frac{1}{6}$. (Dividing by zero cannot be converted into a multiplying problem, which is why we say it has no mathematically agreed answer.)

In Number, Pattern and Calculating 5, children are introduced to a great deal of explicit multiplicative thinking, and two of the basic ideas they have to develop in this connection are those of ratio and of proportion.

Ratio

A ratio is a *multiplicative* comparison (or correspondence) between two values of the same kind, usually expressed in terms of one of the values being 'x times as much as' the other. Most importantly, the 'number of times' does not have to be a whole number, so comparing a 2 cm length with a 3 cm length (see Fig. 1) multiplicatively for example, we could say that 3 cm is '$1\frac{1}{2}$ times as long as' 2 cm, or that 2 cm is '$\frac{2}{3}$ as long as' 3 cm. Conventionally, we could write in ratio notation that they are in the ratio 2:3 or 3:2 to each other, depending on which of them we choose to put first. Children will learn this notation later in their schooling.

Fig 1

Closely connected with the idea of a ratio is that of a rate; a rate is a relationship between two measurements of *different* kinds. Speed is a relationship of distance to time, e.g. 30 miles per (or 'for every') hour. Many (but not all) important rates involve time as one of the measurements because time is such a fundamental aspect of our universe, but currency exchange and interest rates can also be important to us in everyday life.

Proportion

The term 'proportion' is used in more than one way unfortunately, but it is maybe most often used to judge whether two or more sets of values have the same ratio to each other within each set. So, in mixing up two separate glasses of orange squash, if they both have the same strength, i.e. the same ratio of cordial to water as each other, then the two mixtures are said to be in proportion (or proportional) to each other. This leads to a form in which the idea of proportion is often presented as a relationship between four values:

$$\frac{a}{b} = \frac{c}{d}$$

In words, this might be read as, 'the ratio of a to b is the same as the ratio of c to d'. In context, this might come out as, 'the ratio of cordial to water in this glass is the same as the ratio of cordial to water in that glass; the two mixtures are in the same proportion'. Problems involving proportions are commonly situations in which three of the four values are known, and the fourth is to be calculated.

(Notice again how fraction notation is sometimes used to specify ratios. This is because there is an equivalence between saying, for example, 'these two lengths are in the ratio 2:5 to each other', and 'the first length is $\frac{2}{5}$ of the length of the other'. Fractions are often called 'rational numbers'.)

Another use of the word 'proportion' is in contexts where values change in relation to each other; in such situations, a relationship may be either in direct or inverse proportion. In science, mass and weight are said to be *directly proportional* to each other; this means that if you double the mass of something, you will also double its weight. This directly proportional relationship is what allows us to compare masses in practice by comparing weights.

Scaling up or down is another important context within which proportions have to be directly maintained, for example in

recipes, maps and in geometrical transformations. Doubling a distance on a normal map represents double the distance on the actual ground.

In another context, speed and the time needed for a journey are inversely proportional to each other; if you double your speed for a journey, you will halve the amount of time it takes.

Notice how scaling up or down involves multiplying or dividing values by a common factor.

A third use of the word 'proportion' occurs when people ask questions such as, 'What proportion of the tickets were sold to parents?' Two aspects of this use are important: firstly, the words 'proportion' and 'fraction' are interchangeable here; the question could equally well have been, 'What fraction of the tickets …?' And secondly, use of the word 'proportion' usually implies some kind of 'whole' that the proportion is 'of'; ratio comparisons don't imply any 'whole' value, but proportions do.

Finally, it is worth highlighting the fact that proportions are often described and expressed in terms of percentages. This is once again because we are essentially concerned with comparisons here, and percentages were designed to make comparisons easy. We would probably respond to the question above by saying, '70% of all the tickets were sold to parents'.

Connections within multiplicative thinking

In reading the above you will have noticed how much all of these ratio comparisons and relationships depend upon children's underlying skills with and understanding of multiplying and dividing. Since almost all dividing can be converted to an equivalent multiplying (don't forget about the problem with zero), you may appreciate why work on ratio and proportion is collected together within the overall heading of 'Multiplicative thinking'.

Multiplying, dividing, ratios, proportion, fractions, decimals and percentages all link intricately with each other to produce a very varied and complex set of ways to communicate about essentially the same kind of relationships – multiplicative relationships. These relationships, and the ways that we communicate about them mathematically, are crucial to children's future progress in both mathematics and science, as well as in everyday life; they deserve both our and our children's full attention in the Number, Pattern and Calculating 5 activities. 'Converting' between multiplying and dividing, between common fractions and decimals, between fractions and percentages, between improper fractions and mixed numbers and so on is crucial to children's continuing success.

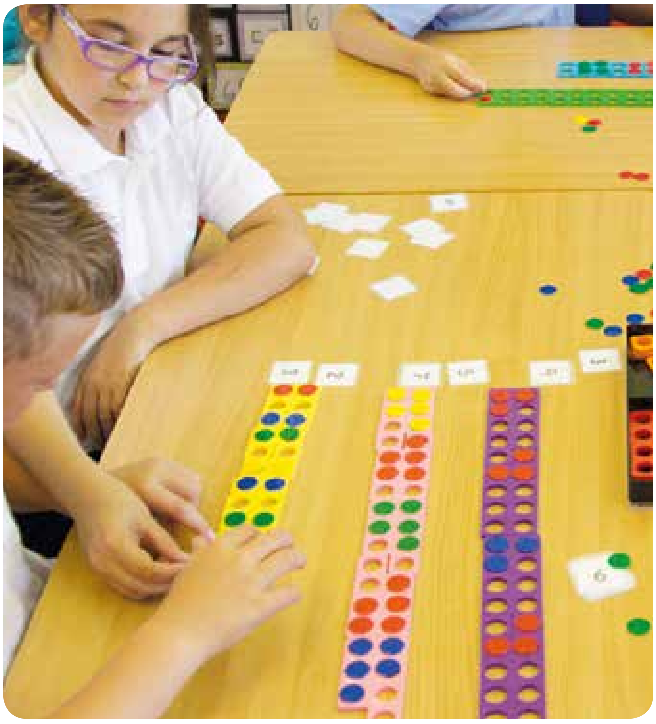
Interestingly, in the 'Mathematics programmes of study: key stages 1 and 2 National curriculum in England 2014', a new term appeared with the aim of collecting together a whole range of problem situations that invite multiplicative thinking; the term introduced was 'correspondence problems', probably because this work will also underlie the significant later development of those other very important mathematical relationships – functions – and functions are essentially about how one variable *corresponds* to another.

In what follows, we offer, and conclude with, some explanation and advice on how to connect a range of problems together that will invite children to think multiplicatively. When offering these types of problem to children, it is important not to suggest or to tell them in advance whether they should multiply, or when to divide; their long-term success depends upon them thinking these things out for themselves.

Correspondence problems

Correspondence problems refer to a variety of situations in which there is said to be a one-to-many correspondence of some kind between two sets of objects. A simple example would be when a number of children are sharing a number of apples; the two sets are the children and the apples. Another example would be when someone has a number of hats and a number of coats and can combine these to make a range of different outfits. In this case, the set of hats 'corresponds' with the set of coats to produce outfits. A third example would be a correspondence between cars and their wheels; each car would correspond to four wheels (usually). Note that, although the correspondence is technically called 'one-to-many', in general the number of elements in each set in a situation can be anything. As a result, such situations are sometimes described as contexts in which n objects correspond to m objects. In practice, there can be the same number of elements in each set ($n = m$), and which set is specified first (the larger or the smaller) doesn't matter.

The point about correspondence problems is that they usually require 'multiplicative thinking'. If one car has 4 wheels, how many wheels will 6 cars have? If there are 3 apples to share equally between 12 children, how much apple will each child get? If a person has 5 hats and 3 coats, how many different outfits can they put together? If I have 48 wheels available, how many cars could I make?



Note that, although the thinking involved in all these different situations is called 'multiplicative', some would lead us to multiply and some situations would probably lead us to divide two numbers.

Essentially, a one-to-many correspondence is a way of describing a ratio, which could also be expressed as n:m or n to m. Such ratios are at the heart of multiplicative situations. In the sharing case of apples and children, the ratio of apples to children is 3:12, and each child gets 'three twelfths' of an apple or $\frac{3}{12}$ (written as a fraction).

When we come to hats and coats, there is no sharing going on, but a 'multiplying' of different possibilities. For every one of three coats, we can put on one of five different hats, giving 1 × 5 possibilities for each coat (ratio 1:5); with three coats the possibilities become (1 × 5) + (1 × 5) + (1 × 5) = 3 × 5 = 15. With wheels and cars, the ratio is 4:1, so there will always be four times as many wheels as cars.

The thing to remember is that 'correspondence problems' are all essentially to do with 'ratio situations', but that we have to think carefully each time whether to multiply or divide to answer a particular question. Later on, as children develop their multiplicative thinking, they will come to understand that all dividing (other than by zero) can be done by multiplying with fractions, and that there's a reason why fractions are called rational numbers. Make sure that children experience a wide variety of correspondence problems and encourage them to think very carefully about what kind of an answer would make sense in each situation.

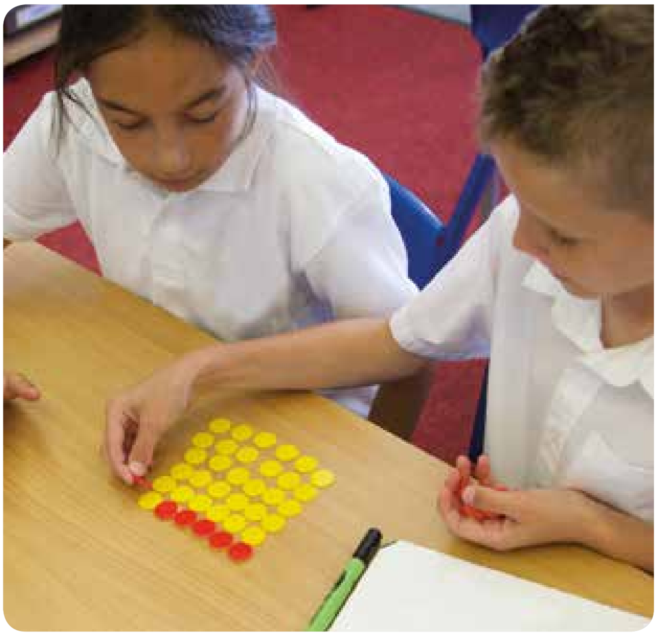
Dr Tony Wing – the theory behind Numicon: what we have learned in our work so far

"Teachers using Numicon quickly find themselves learning from children's responses – as do I and the rest of the Numicon authors.

Numicon is a continually growing understanding of the ways in which structured materials and imagery can be helpful to children in their learning of mathematics.

Using Numicon effectively involves understanding something of the theory behind Numicon, including an understanding of what young children face as they learn how to do mathematics.

The following section sets out what I and the other Numicon authors have learned in our work so far, in order to help with the use of the teaching materials."



Doing mathematics – being active and exploring relationships

It is most helpful to see mathematics as an activity, as something people 'do', actively, rather than as a lot of facts and techniques that have to be passively acquired.

The reason mathematics is thought so important for children in school is that we all want them, after they have left school, not just partly to remember those facts and techniques that they used to pass their exams, but to be able to do mathematics successfully when they later meet new and unfamiliar mathematical challenges in their everyday lives and in their work.

Being able to do mathematics involves being able to pick out key relationships in a situation and then manipulating those relationships to predict outcomes which we are interested in.

For example, in a practical shopping situation, key relationships could include those between prices, totals, budgets, currency structures and cash availability. We could manipulate all these to predict whether we can afford to buy something and, if so, how we could pay. Usually, such a situation would require us to do some calculating along the way. Here, the key relationships would be between numbers themselves: we might be manipulating number relationships by adding, or by finding a difference, to predict a total or an amount of change.

It is worth noting three aspects of doing mathematics in the shopping example:

- Firstly, there is the business of working out which quantities are important to us in the given situation, and how they relate to each other.
- Secondly, there is some calculating to do, with pure numbers.
- Thirdly, there is some interpreting of calculation results in order to predict what will happen when we decide what to do in the practical situation.

Identifying and manipulating key relationships in a situation in order to predict outcomes in this way is often called 'mathematical problem solving', and doing this essentially involves **exploring relationships** (the connections between things) within a situation.

Even a problem as simple as finding out, 'How many children are having lunch today?' involves using some kind of order relationship if we are to predict this successfully. For example, not counting the children in order may mean some are missed or some counted twice.

Numicon constantly encourages children to explore the relationships in situations, to see patterns and regularities, and to use these to make predictions. All this lies at the heart of doing mathematics at any level.

It is worth noting that when working on a real-life problem, such as calculating the cost of shopping, we tend to reach a point where we temporarily forget about the practical context we are in and just work with pure 'numbers'. This challenge of moving backwards and forwards between particular practical situations and an abstract world of numbers presents some of the most significant challenges that children face in learning how to do mathematics.

Of course, some problems crop up only within an abstract world of numbers, for example, as we learn how to calculate more effectively. Yet even these situations require us to **be active** and to **explore** and use the various relationships involved between numbers themselves.

Importantly, doing mathematics in our everyday lives and at work involves working out what to do in situations simply as they crop up; there is no helpfully arranged programme to life (as there is in school), and as adults we have to be able to cope with whatever comes up, in whatever order it appears. Children also have to learn to rise to the challenge of being able to do mathematics in new and often unfamiliar situations, not just try to remember selected techniques that are likely to come up at the end of a period studying a particular topic.

This has important implications for both our teaching and our assessing. Children need to learn how to do mathematics in new and unfamiliar situations, how to actively explore relationships and to manipulate relationships between things in the same ways that mathematicians cope with new situations. Children – in their worlds – are learning to join in with the activity of doing mathematics in fresh fields.[1]

1 For more on this view see Freudenthal, H. (1973) *Mathematics as an Educational Task*. Dordrecht: D. Reidel Publishing Company

Generalizing, thinking and communicating

Doing mathematics makes the everyday world predictable in a surprising number of ways. When we board an aircraft, we expect to arrive safely and at our chosen destination; we expect fresh food to be available to us in our local shops as and when we want it; we expect electricity to flow in our homes and in our workplaces whenever we flick on a switch. How is it that these everyday expectations are met so often in most of our lives? The answer is that aeronautical engineers, navigators, logistics experts, electrical engineers and statisticians predict these things for us by doing mathematics.

Crucially, doing mathematics involves a unique way of **thinking** and **communicating** about situations; a special way of communicating that has been developing especially for the purpose of doing mathematics ever since humans first concerned themselves with quantities and relationships.

Interestingly, our mathematical communicating doesn't just happen between us and other people; we also constantly communicate mathematically with ourselves whenever we do mathematics. We call this thinking. Just try multiplying 481 by 37 and listen to the voice in your head as you work it out; you can hear your own thinking as you communicate your calculating with yourself.

It is important to understand how our thinking and our communicating develop. Building on Vygotsky's work, Sfard argues that our thinking develops as our own 'individualized' version of the communicating that we do with others.[2] The important implication is that children's mathematical thinking is their own individual version of the mathematical communicating they do with their teachers. Learning to join in with the mathematical communicating we use around them is how children learn to think and communicate mathematically for, and with, themselves.

2 Sfard, A. (2008) *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. New York: Cambridge University Press

Making our lives predictable with mathematics and the ways that we communicate mathematically are closely connected. In fact, it is because mathematics aims to predict through seeing patterns and regularities in relationships that mathematical thinking and communicating has developed in the distinctive ways that it has. More specifically, mathematics makes situations predictable through **generalizing**. As a result, what we communicate with and about most often when doing mathematics are **generalizations**.

Even the number we call '3' is a generalization; we want children to understand it as meaning '3 of anything'. The number facts children are expected to remember are all generalizations. We want children to understand that '6 + 2 = 8' means '6 of anything and 2 of anything will together always make 8 things – whatever they are'. In geometry, when we talk about the angles of 'a triangle' adding up to 180°, we mean any triangle, not just a particular one we might have drawn in front of us.

Generalizations are important because they can be used to predict outcomes in particular situations. For example, once we've made the generalization that '4 × 25 = 100', we can predict that: the perimeter of a square of side 25 cm will be 100 cm; if we're given £25 a week for four weeks, we'll have been given £100 in all. We could even use it to calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

In other words, lots of different kinds of particular situations all become much more manageable because of that one generalization.

It is important for children to learn 'number facts' such as '6 × 3 = 18', but the most important thing (if such 'facts' are to be useful) is for children to reach them actively through generalizing themselves, and for children to learn how they can use these generalizations in their mathematical thinking and communicating to make the world they live in more predictable.

If 'generalizing' sounds like quite an abstract and difficult thing to do, try thinking of it as simply looking for patterns. The good news is that human beings are all very good at looking for and finding patterns in our experiences. Young children, in particular, have been phenomenally good at this since the day they were born; they learned to speak their mother tongues simply by being extraordinarily attentive to the patterns in the sounds around them – an incredible achievement.

Numicon taps into this incredible facility children have for spotting patterns in situations – for generalizing – wherever possible. This is at the heart of thinking and communicating mathematically.

How do we communicate mathematically?

Communicating with the sheer density of generalizations that we use when we are doing mathematics is not easy.

First of all, this is because generalizations distance us from the close particulars of our individual lives. This is one big reason why so many people find doing mathematics abstract and remote and – mistakenly – feel it is unconnected to their particular everyday world.

Secondly, it is very difficult to talk, and to think, about anything in general without imagining something in particular. When we want to say something about people in general, we usually have experience with particular people in mind (we only ever meet or hear about particular people); when we want to talk with children about triangles in general, we usually show them one triangle (or perhaps a few) in particular. So it is with number generalizations.

When we begin to talk about the generalization '3' with children, we often show them three things in particular (perhaps three counters), even though we want them somehow to interpret those particular three things as representing 3 of anything. 'Seeing the general in the particular'[3] is at the heart of doing mathematics.

The talking that we do in doing mathematics, both with ourselves and with others, is crucial to our developing mathematical thinking and communicating.

Over the centuries, we have developed sophisticated and effective ways of thinking and talking about numbers and other generalizations in mathematics. Not surprisingly, children can sometimes have trouble joining in with them immediately.

Firstly, in our developed mathematical communicating about amounts and numbers of things in general, we somehow manage to turn our generalizations into objects – mathematical objects. For instance, since it would be very clumsy and awkward to be forever talking about our number generalizations as '6 of anything' and '242 of anything', in practice, we have got used to using a kind of linguistic shorthand and talking about just '6' and '242'. And this has consequences.

In this almost accidental way, we have opted to use number words as nouns and thus to speak and think about the generalizations '6 of anything' and '242 of anything' in the same way as we would if they were actually material objects in the world, like chairs, tables or frogs. We call these generalizations-we-make-into-things, these mathematical objects, numbers. In the previous example, we make the mathematical objects '6' and '242' simply by shortening our generalizing phrases into names.

It is important to realize that by naming something (simply by using a word as a noun), we implicitly announce that what we are talking about is a 'thing', an object; thus just by starting to use number words and symbols as nouns instead of adjectives in our language, we announce to children that in talking about numbers we are talking about things of some kind.

It is also important to remember that, in using number words as nouns, all we do is 'invent' numbers within our use of mathematical language – we don't actually make anything that is real in a world beyond language. A generalization (such as a number) is not a material thing; it is a thought formed in a way of using words (or symbols), and we start talking about such odd things very early in our work with children.

More curiously, most of us who have long ago learned to 'handle' numbers tend to feel as if we literally move them around and connect them as we do calculations, either on paper or 'in our heads' – again as if numbers had the properties of physical things. As numerate adults, we commonly calculate with symbols on a page, or on a calculator display, as if the numerals were somehow the abstract number ideas themselves.

As an example, try dividing 273 by 46 and see if you don't feel as if you are treating the numerals involved as if they are number objects of some kind, and that you just move them around and exchange some for others according to the particular rules you have learned. Don't numbers become just 'things on paper' for you, or numerals in your head or on a calculator display, as you work?[4]

3 Mason, J. & Pimm, D. (1984) *Generic Examples: Seeing the General in the Particular, Educational Studies in Mathematics*, 15(3) p277–290

4 Significantly, most of us use imagery of various kinds as well as numerals, as we calculate. But when asked to imagine the number 'ninety four', for example, most of us picture two numerals ('94') – in that order – regardless of whether there is also an image such as a number line involved as well.

As experts, we have come to think and talk about the generalizations we call numbers as things – as objects – as if these mathematical objects were the same kind of things as physical objects that we meet in our everyday worlds. We handle them, we move them around and we line them up on a page. In general, as we do this, we tend to use numerals as if they were the number 'things' that we have invented – simply, remember, by changing how we use number words. How is it possible for children to make sense of all these invisible mathematical 'objects' that we suddenly start talking about in association with numerals in their schooling?

The first thing to observe is that too many children don't ever really make sense of the number objects they meet in their schooling. In practice, we often just expect very young children to move smoothly from talking about 'three sweets' or 'three pencils' (using number words as adjectives referring to physical objects) to talking about just '3' (the same word now used as a noun without any accompanying referents) – as we do.

A little later on, as we introduce fractions, we expect children to move from talking about 'half a pizza', or 'half a bar of chocolate', to talking about just $\frac{1}{2}$ as a thing that is not 'a half of' anything in particular.

This must all seem very strange, and most young children do not so much understand what we are doing as just 'try to go along with it'.

What do we show children as we talk about 'numbers'?

Confusingly for children, at the same point as we start talking about these generalizations (i.e. numbers), using nouns for them as if they were material things, we also commonly start focusing heavily on recording with numerals – as if, by coincidence, the written marks children can actually see on the board or the page are the numbers we are talking about.

There's a good reason for the move to symbols in mathematical communicating at this point: we can only talk about generalizations with what Bruner called symbolic representation.[5] We cannot draw a picture to show '6 of anything' because as soon as we make a picture, or count out 6 physical objects, we are showing children '6 of something'. It is symbolic representation (i.e. words and numerals) in particular that allows us to communicate most readily about invented, non-material things – in this case, pure numbers.

However, unless we are careful, children will quite reasonably – and commonly do (subconsciously) – think that the visible numerals we are showing them actually are the mysterious number things that we have now begun talking about.

The English language is not very helpful either, since, in English, we call numerals 'numbers' as well. We talk about 'the number of that bus' or 'the number of your house' when we are referring to numerals that we can see. In class, we ask children to 'write down' numbers while expecting them to draw numerals.

Importantly, at other times we also want children to understand that '3 and 3 are equal to 6'. What sense can that make to a child who thinks that the numerals '3' and '6' they are looking at are the numbers the teacher is talking about? It is very hard to make sense of such number relationships if all we have to communicate and to think with are number words and symbols; '3' and '3' together don't look like '6'– they look more like '33'.

It is our challenge to find ways of communicating about mathematical objects children can't see (pure numbers) in ways that avoid confusing them with numerals, and which also allow us all to explore relationships between these invisible things.

One key solution is to bring real physical objects and imagery into our communicating; to import special ways of **illustrating**, enactively and visually, relationships between the mystical generalizations we are talking about.

It is for this reason that, when introducing children to numbers, we commonly prepare for and supplement their use of numerals with a range of physical objects and imagery, using actions with objects and visual illustrations to help children 'see' and 'feel' how the invisible number objects we invent with our words relate to each other.[6]

Crucially, this is the point at which we need children to start 'seeing the general' in the particular enactive and visual illustrations that we offer. We can help children a great deal with this in the discussions that we have with them as they work. If we use counters, or cubes, or beads, or beans to talk about numbers, we want children to focus only on 'how many' there are, and to ignore the kinds of physical objects

5 See Bruner, J. (1966) *Towards a Theory of Instruction.* Cambridge, MA: Harvard University Press

6 In this, we are combining Bruner's three modes of representation – enactive, iconic and symbolic – together in order to enrich children's learning as much as possible.

we are using. We need children to stress how many discrete objects are before them, and to ignore what sort of objects they are. 'Stressing and ignoring' lies at the root of 'seeing the general in the particular' illustration, and our conversations with children as we 'illustrate' are crucial.[7] We need to keep asking, 'Would it make any difference to our calculation if those counters were beans? Or what if they were pencils?'

In Bruner's terms, since by thinking and communicating mathematically we are working with generalizations (this is how mathematics helps us to predict), we will eventually do this most effectively by using symbolic representation, e.g. by using numerals and words in our writing and talking to represent our number generalizations.

However, numeral symbols and words are merely conventionally agreed marks on a page and sounds that we hear and – crucially – are dependent for their interpretation upon the prior and accompanying experiences of both action and imagery that led up to the generalizing they symbolize.

Thus, effective use of numerals by children (symbolic representation) in their thinking and communicating – in other words, the calculating we want them to be able to do – is dependent upon their prior and accompanying use of enactive and iconic representation in their experiences with numbers of things.

Children's learning to think and communicate mathematically with generalizations will eventually lead them to mastery of associated symbolic representation (numerals and words). In order to reach and sustain that mastery however, children's route lies necessarily through use of enactive and iconic representation (action and imagery). The most effective teaching therefore involves children 'individualizing' the actions, imagery, words and symbols we use, and joining us in using them in the mathematical communicating of their classrooms.

It is also important to remember that most effective mathematical thinking and communicating at all levels involves a rich blend of enactive, iconic, and symbolic representation together. Action and imagery always support the interpretation of mathematical symbols and there is no point at which children should be expected to leave actions and imagery (physical illustration) behind to do 'grown up' thinking. As children's thinking develops, they 'internalize' the actions and imagery that have led to their effective use of symbols, but physical materials and imagery should always be available in classrooms for children to call upon as new ideas are met, and familiar ones reviewed.

To sum up this part of the theory behind Numicon, numerals (and words) are vitally important symbols that gradually become ciphers for the generalized number objects used in advanced mathematical communicating and thinking.

However, as arbitrary conventional symbols, numerals and words cannot illustrate any number relationships. If children are to learn how to handle number generalizations and their connections effectively, they need to have ways of mediating their mathematical communicating (and thus mathematical thinking) with illustrations that will help children to 'see the general in the particular'.

Sfard (op. cit.) calls the objects and images used for this illustrative purpose communication mediators, since they are used to mediate communicating with children. Such objects and imagery subsequently come to mediate children's mathematical communicating with themselves – their thinking about numbers.

Numicon introduces children to thinking and communicating about numbers with a combination of numerals and words in writing and talking, and by mediating this symbolic communicating with physical objects and imagery (enactive and iconic communicating) to illustrate the generalizing that both invents 'numbers' and establishes the relationships between them.

Which communication mediators should we use? Does it matter?

As noted previously, doing mathematics centres around generalizing and using generalizations. In learning how to do mathematics, we become capable of solving an increasing variety of mathematical problems as we call upon an increasing range of generalizations from our past experiences.

Young children usually first learn to generalize about quantities and amounts and talk about number generalizations in their communicating with us, and they then call upon these early generalizations when they later

7 For helpful discussion on this and related views see Mason, J. & Johnston-Wilder, S. (Eds) (2004) *Fundamental Constructs in Mathematics Education*. London: Routledge Falmer. p126ff

need to study, and to calculate with, relationships between quantities and amounts in ever more complex situations.

Children's facility in 'handling' the generalizations we call numbers – their ability to calculate with whole, positive numbers, fractions, negative numbers, for example – is fundamental to almost all of their subsequent progress with mathematics.

We also note that working with numbers – calculating – takes children into a world of invented objects that, although not real, are very significantly connected. Unless children can also begin to generalize about number connections – to generalize about their number generalizations – their calculating will remain very primitive. Typically, children who fail to make much progress with calculating remain restricted to the use of laborious and very basic counting procedures. Fortunately, the regular and systematic relationships between the numbers we invent both makes generalizing about them possible, and also gives us a clue as to which illustrations are most likely to be helpful to children exploring them. Numbers are invented in well-organized systems, and our illustrating needs to reflect their systemic relationships.

In essence, in order to calculate effectively, children need to explore the relationships of numbers to each other. In other words, they need to explore the various ways in which the generalizations of our mathematical communicating about quantities connect with each other. For instance, the ways in which numbers are ordered is important, as are equivalences such as '6 + 2' being equivalent to '8'. The fact that it doesn't matter which way round we add two numbers together, they are still equivalent to the same number total, is important. As is the fact that if we 'add' more than two numbers together it doesn't matter what order we do that in. The numbers '0' and '1' seem to be peculiar in that 'adding 0' and 'multiplying by 1' seem to have no effect at all. Adding seems to be the opposite of subtracting, while multiplying seems to be the opposite of dividing. Interestingly, adding and multiplying seem to be closely connected with each other, as do dividing and subtracting. These are all generalizations about how 'numbers' relate to each other, and they are crucial to the effectiveness of children's calculating.

Scattered (unstructured), random collections of loose objects as illustrations render number relationships, such as those outlined, obscure and are only really useful as opportunities for children to impose relationships upon. Such unorganized collections of, for example, cubes and counters, initially offer opportunities for early counting practice. However, if, as illustrations, they remain unorganized, they are a poor foundation for calculating. It is very difficult to mediate any communicating about number relationships with unorganized collections.

Numicon uses objects and imagery specifically to mediate communicating about number relationships. Numicon brings physical objects and imagery into mathematical communicating that illustrate, above all else, the ways in which numbers are connected. When random collections of discrete objects are introduced in Numicon, children are always expected to put order upon them: to make relationships in what they see.

Numicon also uses a variety of actions, physical objects and imagery because children need to be generalizing from their experiences and conversations and children can only generalize from a variety of experiences.

Number lines of various kinds are used to foreground the order relationships of numbers and Numicon Shapes to give regularity, pattern and two-dimensional shape to number relationships.

Number rods are used to allow children to relate the sizes of numbers to each other in many more ways than are possible with number lines.

Loosely arranged collections of objects are only introduced to offer children important opportunities to impose their own relationships upon such situations: both order structures (when they count) and shape regularities (when they ‘find how many’ without counting).

By choosing a range of physical materials and imagery suited particularly to illustrating relationships, Numicon offers children the crucial enactive and iconic experiences that enable them to manipulate symbolic representations of their generalizations (numerals and words) effectively in their calculating.

The importance of context

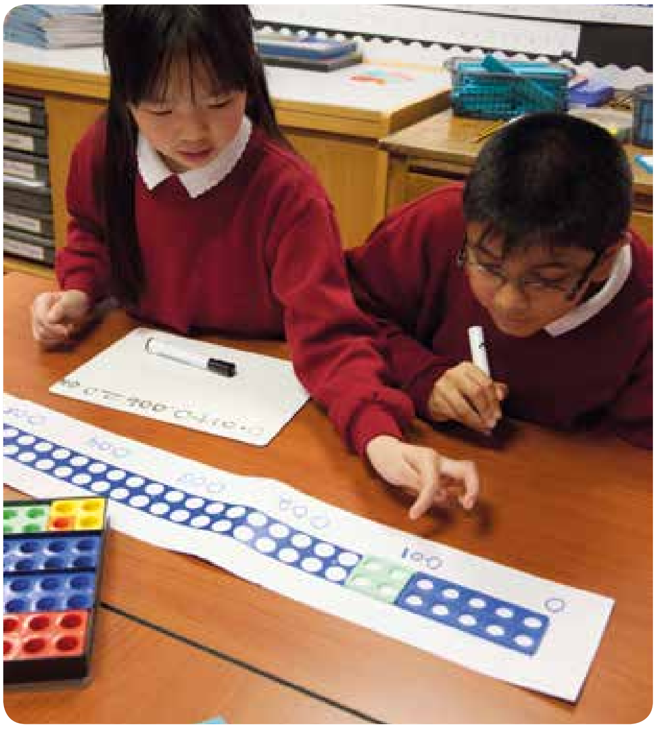
We began this discussion by noting that we want children to leave school able to do mathematics as required in their everyday lives and in their work. In other words, we want children to be able to solve new and unusual mathematical problems when they meet them. Children thus need to be able to explore and pick out key relationships in new and unfamiliar situations, and manipulate them in order to render those situations predictable.

Usually, doing mathematics also involves children moving into the abstract world of number generalizations at key stages – pure calculating – before returning to the practical situation with one or more numbers which need interpreting in the particular context of the problem.

Numicon teaching materials advocate approaching children knowing when the generalized facts and techniques of mathematical communicating are useful through contexts and talking. Within Numicon, each group of activities begins with a carefully chosen particular context, in which the mathematics that is to be learned would be found useful.

Work begins on an activity group by talking about the relationships within which a need for some kind of mathematical response is established; children discuss their initial responses to the questions and challenges involved before moving to the generalizing mathematics to be learned through those activities.

In associated practice activities, children have opportunities to use the general mathematics they are learning in further particular contexts, thus learning from further opportunities to judge when such mathematical generalizing can help.



In Explorer Progress Books tasks, children are presented with challenges that invite them to use mathematics they have been learning, but in unusual and unfamiliar contexts. As their name implies, these tasks invite children to explore relationships in fresh fields.

Summarizing

The key to understanding Numicon is to recognize that doing mathematics involves learning how to **communicate mathematically**, and that mathematical communication is essentially about **generalizations**: it is through generalizing in mathematics that we make our particular worlds predictable.

In the process of doing mathematics, we often make our generalizations into things that we cannot see: *mathematical objects*.

Communicating is at the heart of doing mathematics because that is where our *mathematical objects* – our generalizations – are made.

However, communicating with generalizations is not easy; to help children move between the worlds of mathematical generalizations and of particular situations, we will always need to illustrate our communicating by **being active**, by **illustrating**, and by **talking** as children **explore relationships** they can physically see and feel.

Glossary

Most mathematical terms used in this *Implementation Guide* and the *Number, Pattern and Calculating 5 Teaching Resource Handbook* can be found in a good mathematics dictionary such as the *Oxford Primary Maths Dictionary*.

Other terms you might not be familiar with are explained in this glossary.

base-ten apparatus

A set of concrete materials, systematically designed to help children understand our place value system. Small cubes, sticks of 10 cubes, flat squares of 100 cubes, and large cubes of 1000 small cubes are used when talking about ones, tens, hundreds and thousands respectively. (See Fig. 1.)

bead string

A string of coloured beads (usually red and white, but not necessarily) arranged so that successive decades of beads are alternately coloured (see Fig. 2). There is enough free space available on the string for beads to be moved backwards and forwards according to the numbers and calculation conversations being mediated by use of the bead string.

bridging across a multiple of 10 when adding or subtracting

Bridging is a calculating technique that involves partitioning (splitting) the number to be added or subtracted. Bridging can be used across any number, but bridging across multiples of 10 is especially useful, since this exploits the very basic adding and subtracting facts to 10 that children learn early on, e.g. $8 + 9 = (8 + 2) + 7 = 17$ (see Fig. 3).

Bruner

Jerome Bruner (1915–2016) was an extremely distinguished and influential psychologist. His distinctions between enactive, iconic and symbolic representation have been particularly influential in the field of education.

column value

Numbers are often arranged in columns, with each column having a place value, e.g. hundreds, tens or ones. The numeral '2' in '327' is said to have a column value of two tens.

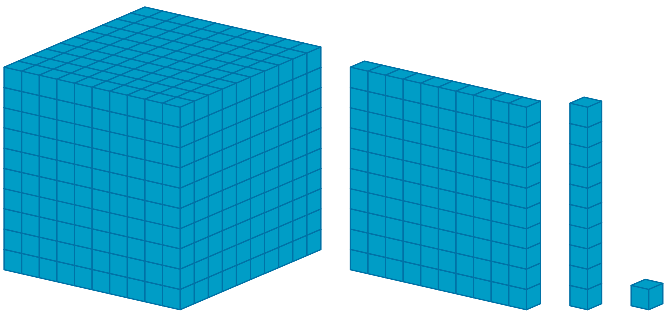
(See also **quantity value**.)

communication mediator

A communication mediator is an object, image or action that helps communication in some way. These communication mediators need to be carefully introduced to children, e.g. Numicon Shapes and number rods become communication mediators when they help to illustrate discussions about the relationships between numbers; number lines, graphs and bead strings can all act as communication mediators. However, these things only become communication mediators if they are used to support communication. Any physical object or image is just a physical object or image unless it is actually supporting communication; there is 'no magic in the plastic'.

enactive, iconic and symbolic representation

Jerome Bruner (1966) distinguished three key ways in which we humans represent experience to ourselves: through enactive (internalized action), iconic (sensory) and symbolic (forms of language) representations. In Numicon we seek to combine all three forms of representation so that children experience number ideas through action, imagery and conversation.



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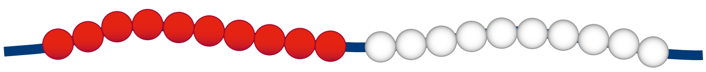

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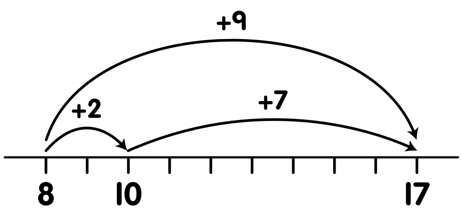

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enumerate

To name how many distinct objects there are in a collection. In Numicon, this term is used in activities that focus on finding how many without counting each individual object; children do this by making Numicon Shape patterns.

generalization

A statement or observation (not necessarily true) about a whole class of objects, situations, or phenomena. Generalizations are essential and everywhere in mathematics, and for this reason children need to generalize, and to work with generalizations, constantly. Numbers are generalizations, as are rules about numbers, such as, 'it doesn't matter which way around you add two numbers, you will always get the same answer'.

number fact

The term 'number fact' usually refers to an operation on two or more numbers, together with its outcome. So '6 + 3 = 9' is a 'number fact', as is '256 ÷ 16 = 16'. In UK schools, these are often referred to as 'number bonds'.

number names/objects/words

Adults commonly talk about numbers as if they are objects, i.e. we often use number words such as 'four', or 'twenty-three', as nouns; we ask children questions such as, 'What is seven and three?' In our language, nouns name objects, so we all commonly (and unconsciously) assume that if we use a word as a noun it must be naming an object. So when we use number words as nouns we assume they must be being used to name number objects – thus, according to the way we use words, numbers are often treated as if they are objects.

It is important to remember that we do not always use number words as nouns; quite often we use those same words as adjectives, as in, 'Can you get me three spoons?' One of the key puzzles for children to work out is how (and when) to use number words as adjectives and when as nouns.

number sentence

The metaphor of sentence (from the use of the word 'sentence' in literacy and grammar) is sometimes used to refer to the writing of a number fact in horizontal form, from left to right. So, 4 + 23 = 27 is a number sentence because it is written in the same graphical manner as a normal written sentence in prose.

number trio

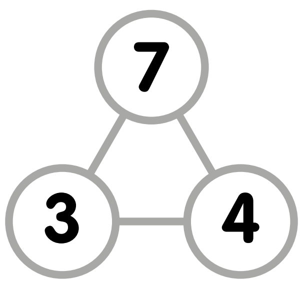
A number trio describes a set of three numbers that relate inverse adding and subtracting facts, e.g. 3, 4 and 7 (see Fig. 4). These are used in Numicon, together with specific forms of illustration, to support children's development of adding and subtracting number facts.

numerals

Numerals function as shorthand for number words. The numeral '5' is shorthand for the word 'five'. When used as a noun, the numeral '5' is often said to represent or stand for the number object we call 'five'.

Numicon Shape pattern

When we refer to a Numicon Shape pattern we are referring to the system of arranging objects or images (up to 10 in number) in pairs alongside each other that is sometimes called 'the pair-wise tens frame'. Fig. 5 shows the Numicon 7-pattern.



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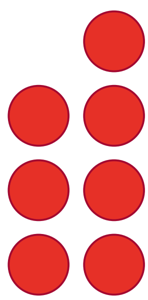

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Numicon Shape

Numicon Shapes are pieces of coloured plastic with holes (ranging from 1 hole to 10 holes) arranged in the pattern of a pair-wise tens frame (see Fig. 6).

Piaget

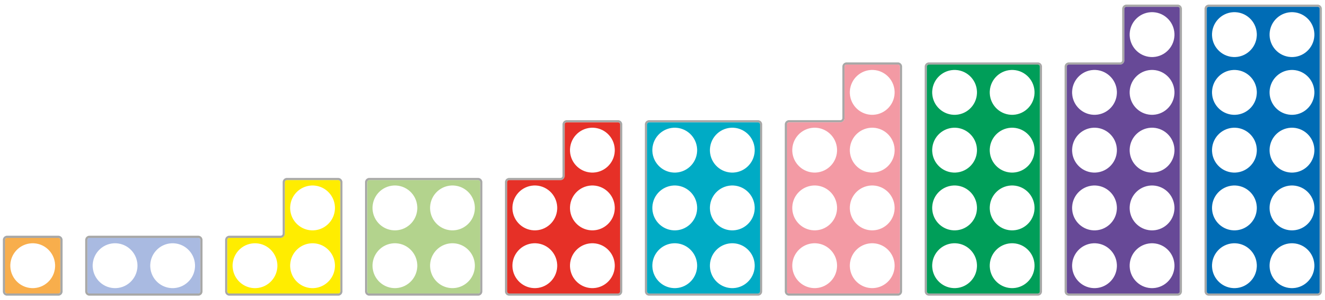
Jean Piaget (1896–1980), was a philosopher and psychologist who spent years studying young children and how they learn. He believed that the development of knowledge is based on developing cognitive structures and that children should be actively involved in their own learning.

quantity value

The numeral '2' in '327' is said to have a quantity value of 20 (twenty). (See also **column value**.)

Vygotsky

Lev Vygotsky (1896–1934) contributed a uniquely social dimension to the study of children's thinking. In particular he stressed the role of expert adults in supporting a child's new learning; this occurs optimally in a child's 'zone of proximal development'. Importantly, he saw the development of a child's thinking as crucially influenced by what he characterized as the 'internalization' of speech.



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