numicon®

Number, Pattern and Calculating 6 Implementation Guide

Written and developed by

Jayne Campling, Romey Tacon and Dr Tony Wing

OXFORD

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford, OX2 6DP, United Kingdom

Oxford University Press is a department of the University of Oxford. It furthers the University's objective of excellence in research, scholarship, and education by publishing worldwide. Oxford is a registered trade mark of Oxford University Press in the UK and in certain other countries.

First Edition published in 2016

ISBN 978-0-19-848980-1

10 9 8 7 6 5 4 3 2 1

Typeset by Phoenix Photosetting, Chatham

Paper used in the production of this book is a natural, recyclable product made from wood grown in sustainable forests. The manufacturing process conforms to the environmental regulations of the country of origin.

Printed in China by Hing Yip Printing Co. Ltd

Acknowledgements
Written and developed by Jayne Campling, Romey Tacon and Dr Tony Wing

Cover photograph by Earl Smith

Photographs by Earl Smith, except: inside cover: Jonty Tacon; pp. 5–7: Chris King; pp. 17, 28, 39, 42, 54, 59–61, 63, 65: Suzy Prior; pp. 15, 36, 50, 51, 53, 58, 64: Damian Richardson

With special thanks to: St Joseph's Catholic Primary School, Convent Garden; Telscombe Cliffs Community Primary School, Peacehaven; Francis Askew Primary School, Hull; Holy Trinity Catholic Primary School, Wolverhampton; Kibworth CE Primary School, Kibworth; The Blake CE (Aided) Primary School, Witney; West Park C.E. (Controlled) First and Middle School, Worthing; Peacehaven Infant School, Peacehaven; Meridian Community Primary School, Peacehaven

The authors and publisher would like to thank all schools and individuals who have helped to trial and review Numicon resources.

www.oxfordprimary.co.uk/numicon

About Numicon

Numicon is a distinctive multi-sensory approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Numicon was founded in the daily experience of intelligent children having real difficulty with maths, the frequent underestimation of the complexity of the ideas that young children are asked to face when doing maths and recognition of the importance of maths to them and to society as a whole.

Numicon aims to facilitate children's understanding and enjoyment of maths by using structured imagery that plays to children's strong sense of pattern. This is done through research-based, multi-sensory teaching activities.

Numicon takes into account the complexity of abstract number ideas and seeks to foster the self-belief necessary to achieve in the face of challenge or difficulty.

Through the combination of communicating mathematically (being active, talking and illustrating), exploring relationships and generalizing, children are given the support to structure their experiences: a vital skill for both their mathematical and their overall development.

A multi-sensory approach, particularly one that makes use of structured imagery, provides learners with the opportunity to play to their strengths, thereby releasing their potential to enjoy, understand and achieve in maths. By watching and listening to what children do and say, this enjoyment in achievement is also shared by teachers and parents.

Numicon strives to support teachers' subject knowledge and pedagogy by providing teaching materials, Professional Development and ongoing support that will help develop a better understanding of how to encourage all learners in the vital early stages of their own mathematical journey.

Contents

Welcome to Number, Pattern and Calculating 6

Before you start teaching, take some time to familiarize yourself with the Number, Pattern and Calculating 6 starter apparatus pack, the teaching resources and the pupil materials, to help you and the children get the most out of using Numicon.

Use this Implementation Guide:

- to find out more about what Numicon is
- to find out how using Numicon might affect your mathematics teaching
- to learn about the key mathematical ideas children face in the activity groups
- for more detailed information on the theory behind Numicon from Dr Tony Wing.

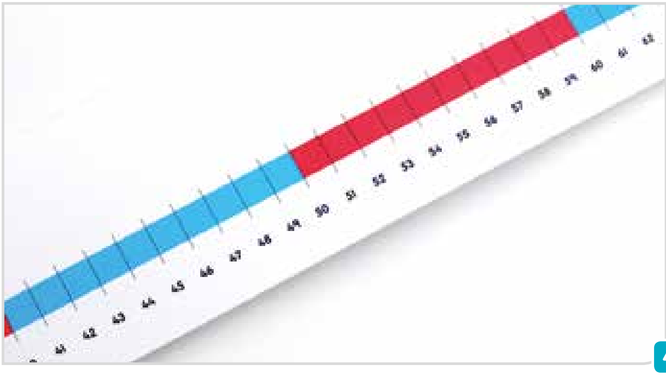
You will find extra support for teaching, planning and assessing using Numicon in the Numicon Planning and Assessment Support available on www.oxfordowl.co.uk.

You will find more information, videos, ideas and free resources on the Numicon website: www.oxfordprimary.co.uk/numicon. Here, you can sign up to receive the latest Oxford Primary news by email.


1


2


3


4

What's in the Numicon starter apparatus pack?

The following list of apparatus supports the teaching of Number, Pattern and Calculating 6. These resources should be used in conjunction with the focus and independent activities described in the activity groups.

Starter apparatus pack contents

- Numicon Shapes – box of 80 (× 2)
- Numicon Coloured Counters – bag of 200 (× 2)
- Numicon Baseboard Laminate – set of 3 (× 2)
- Numicon 0–100 cm Number Line – set of 3 (× 2)
- Numicon 1 000 000 Display Frieze
- Numicon 0–1·01 Decimal Number Line
- Numicon ¯12–12 Number Line
- Numicon 10s Number Line Laminate (× 4)
- Numicon Fraction Number Line Laminate
- Numicon Spinner (× 4)
- Numicon 0–100 Numeral Cards
- Numicon 1–100 Card Number Track (× 3)
- Number rods – large set
- Numicon 1–100 cm Number Rod Track (× 3)
- Extra Numicon 10-shapes – bag of 10 (× 3)
- Numicon Feely Bag (× 2)
- Magnetic strip

Numicon Shapes 1

These offer a tactile and visual illustration of number ideas. The Shapes are also a key feature of the *Numicon Software for the Interactive Whiteboard*, useful for whole-class teaching sessions. However, the Software is not a substitute for children actually handling the Shapes themselves. It is strongly recommended that children are provided with their own individual set of Numicon Shapes 1–10 for use in whole-class sessions.

Numicon Coloured Counters 2

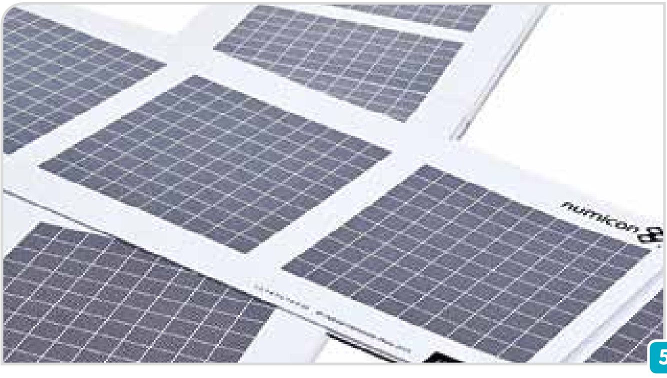
These red, yellow, blue and green Counters can be used for building arrays when multiplying and dividing, for arranging into Numicon Shape patterns in counting activities, and for exploring patterns and possibilities. Using them with the baseboard laminates allows children a clearly defined field of action upon which to create their arrays or patterns.

Numicon Baseboard Laminate 3

This double-sided laminated square baseboard is an empty 100 square, scaled to take Numicon Shapes and Counters. The white side is used in many activities, providing a defined field of action for number and pattern investigations. The orange side is a decimal baseboard laminate which can be used to help children continue to explore decimal numbers up to two decimal places. It offers a possible representation of an expanded 1-shape for children to represent decimal parts of numbers on.

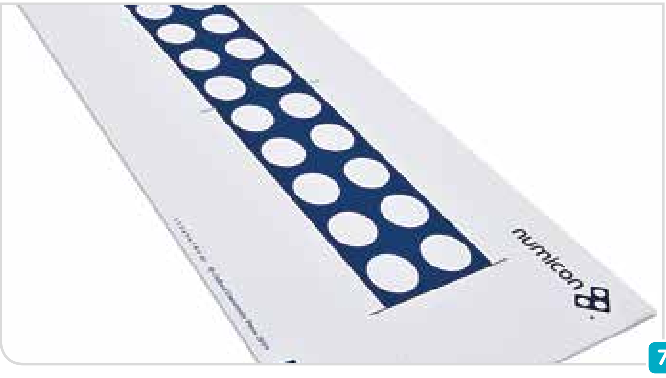
Numicon 0–100 cm Number Line 4

The points on this number line are 1 cm apart and are labelled from 0–100. The number line is divided into decade sections, distinguished alternately in red and blue, to help children find the 10s numbers that are such important signposts when children are looking for other numbers. This resource can also be used with number rods.

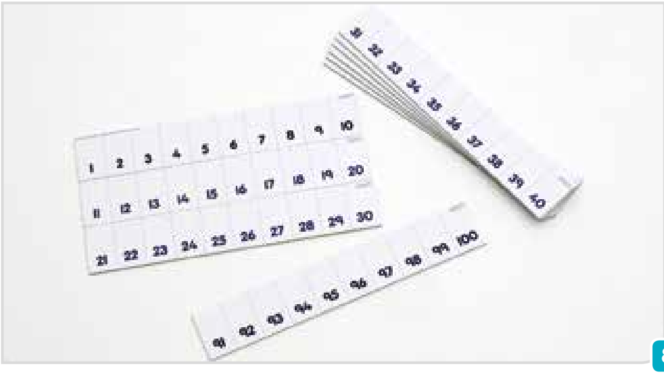


5

6



7


8

Numicon 1 000 000 Display Frieze 5

This display frieze shows 100 blocks of 100 squares, each made up of 100 dots helping children recognize 1 000 000 as a cube number. The sections can be arranged end-to-end horizontally or as an array such as in a square of 10 × 10 blocks, helping children also recognize 1 000 000 as a square number. It provides a visual reference point for the scale of large numbers as well as supporting discussions about the application of square and cube numbers to area and volume.

Numicon 0–1·01 Decimal Number Line

The points on this number line represent thousandths, hundredths and tenths. Zero and one are labelled while the other points are left blank to encourage children to think about what each point shows. Children could record decimals in tenths, hundredths or thousandths on this number line.

Numicon $^{-}$12–12 Number Line 6

This number line shows negative and positive numerals. It provides a visual reference for counting and calculating with negative numbers and can be displayed on the wall or given to children for use on their tables.

Numicon 10s Number Line Laminate 7

This laminated number line, scaled to take Numicon Shapes, shows Numicon 10-shapes laid horizontally end-to-end with points marked, but not labelled. The points can be labelled with multiples, fractions, decimals, percentages or negative numbers, using a whiteboard pen, to support children with exploring number relationships and calculating.

Numicon Fraction Number Line Laminate

This laminated number line starts at zero and has fifty unlabelled points. The points can be labelled with any denomination of fraction children are working with. This provides a valuable tool for exploring equivalences and comparing fractions with different denominators.

Numicon Spinner

The Numicon Spinner can be used in many practice activities. Different overlays (provided as photocopy masters) can be placed on the spinner to generate a variety of instructions for children to follow, including: numerals, percentages and symbols of arithmetic notation. The spinner also features on the *Numicon Software for the Interactive Whiteboard*.

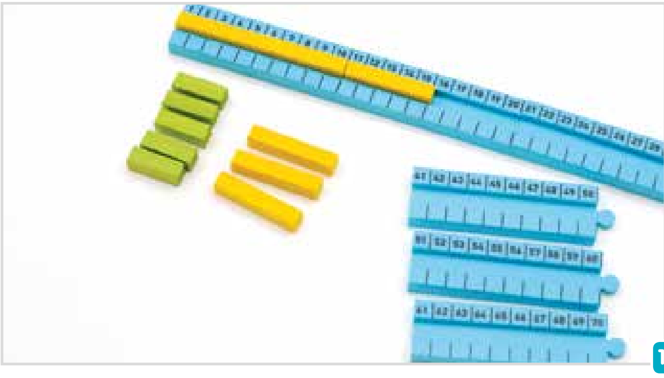
Numicon 0–100 Numeral Cards

This pack of 0–100 numeral cards can be used in several activities for generating numbers which children then work with. It is also used in some of the whole-class and independent practice activities and games.

Numicon 1–100 Card Number Track 8

This number track is divided into ten strips, numbered 1–10, 11–20, 21–30 and so on. The sections can be arranged horizontally end-to-end as a number track, or as an array similar to a 100-square.

9



10

11

12

Number rods 9

A box of number rods contains multiple sets of ten coloured rods, 1 cm square in cross section. The shortest is 1 cm in length, the longest 10 cm. These offer another structured illustration for number and are used alongside Numicon Shapes in many of the activities. Being centimetre-scaled, they can also be placed along the Numicon 0–100 cm Number Line.

Numicon 1–100 cm Number Rod Track 10

Use this for teaching about place value, partitioning, multiplying and dividing. The decade sections click together into a metre-long track. Designed to take number rods, it can be separated easily into sections and arranged as an array.

Numicon Feely Bag

Children use the Feely Bag to generate numbers for problems and to explore the concept of 'unknown' numbers in algebra.

Magnetic strip

This self-adhesive magnetic strip can be cut into pieces and stuck onto the back of Numicon Shapes or number rods so that they can be used on a magnetic whiteboard.

Available separately

Numicon Software for the Interactive Whiteboard 11

This rich interactive tool is designed for use with the whole class to introduce key mathematical ideas. It includes: number lines featuring Numicon Shapes, the Numicon Pan Balance, objects for counting, coins, Numicon Spinners and much more.

Individual sets of Numicon Shapes 1–10

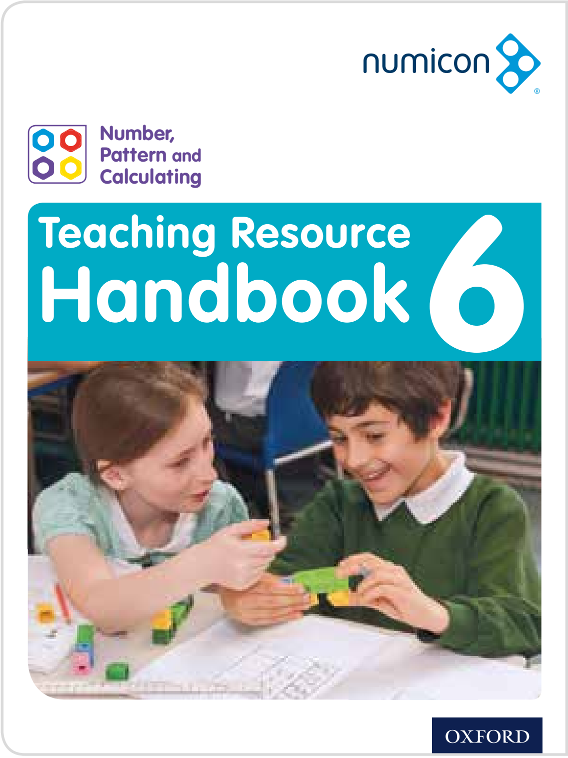
These are designed for multi-sensory whole-class lessons, where each child has their own set of Shapes and is encouraged to engage with them. They are especially useful when used in conjunction with the *Numicon Software for the Interactive Whiteboard* to help teachers assess children's individual responses from the Shapes children hold up.

Numicon Pan Balance 12

Using Numicon Shapes or number rods in this adjustable Pan Balance enables children to see equivalent combinations, helping them to understand that the '=' symbol means 'is of equal value', thus avoiding the misunderstanding that it is an instruction to do something. Children can easily see which Shapes are in the transparent pans. A virtual balance is also featured on the *Numicon Software for the Interactive Whiteboard*.

Other equipment

Some activities use apparatus found in most classrooms, e.g. **sorting equipment, base-ten apparatus** and **interlocking cubes** as well as real-life items such as receipts and clothing labels. Opportunities to use these are highlighted in the '**have ready**' sections of each focus and independent practice activity.



13

14

What's in the Numicon teaching resources?

Number, Pattern and Calculating 6 Teaching Resource Handbook 13

This contains thirty activity groups clearly set out and supported by illustrations. Each core activity group begins with the educational context, learning opportunities, assessment opportunities and important mathematical vocabulary that support mathematical communicating in the activity group. To support teachers' assessing of children, there are notes on what to 'look and listen for' as children work on the activities, as well as suggestions for whole-class and independent practice. Photocopy masters supporting the thirty activity groups are included at the back of the Teaching Resource Handbook.

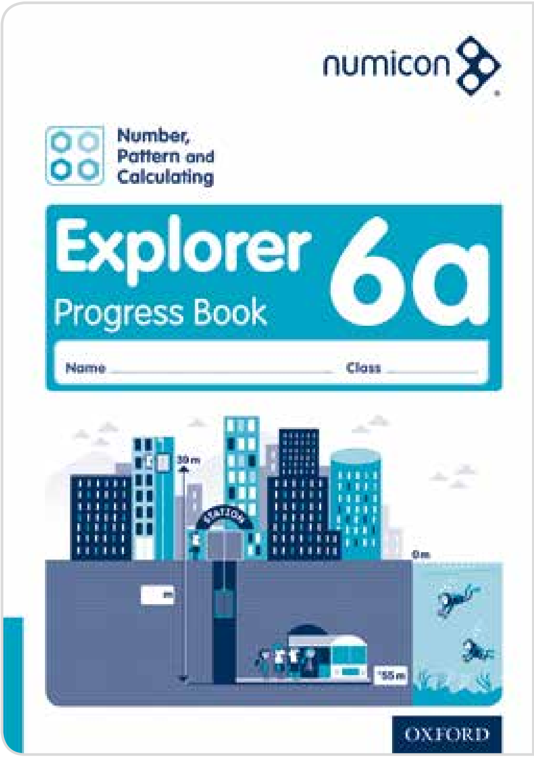
Support for planning and assessment is included at the front of the handbook. There you will find:

- information on how to use the Numicon teaching materials and the physical resources
- long- and medium-term planning charts that show the recommended progression through the thirty activity groups
- milestones to help assess how children are progressing in their learning
- an overview of the activity groups.

Number, Pattern and Calculating 6 Implementation Guide 14

This introduces what Numicon is and how Numicon helps children meet the demands of learning mathematics.
It also includes some practical advice about what to do when preparing to teach with Numicon and answers some key questions about how to use Numicon in practice.
The 'Key mathematical ideas' section provides useful explanations about the important mathematical ideas children will meet in the thirty activity groups of the Teaching Resource Handbook and how these ideas develop when teaching with Numicon. There is also a further chapter with more background detail on the research that inspired Numicon and the rationale behind the pedagogy.

The different sections of the Implementation Guide can be accessed as and when necessary to best help you with your teaching.



15

16

Number, Pattern and Calculating 6 Explorer Progress Books 15

The Explorer Progress Books offer children the chance to try out the mathematics they have been learning in each activity group. In children's responses, teachers will be able to assess what kind of progress individual children are making with the central ideas involved in each activity group.

It should be stressed that the challenges in the Explorer Progress Books are not tests. There are no pass/fail criteria; the challenges are simply designed to reveal how well children can use the mathematics they have been learning in a new situation.

Being able to use mathematics in an unfamiliar situation is a significant indicator of children's understanding. Many of the tasks from the Explorer Progress Books set mathematics in a new or different context and, where possible, the challenges are set in an open way, inviting children to show how they can reason with the ideas involved rather than testing whether they have learned a routine solution to routine tasks.

As with children's classroom activity work, a range of apparatus should be freely available to children as they reason with the ideas in their Explorer Progress Books.

In addition, there is also scope for self-assessment in each Explorer Progress Book in the form of a Learning Log, which can be used flexibly throughout a term, or to summarize the learning of a block of work.

Three books are provided for the thirty Number, Pattern and Calculating 6 activity groups.

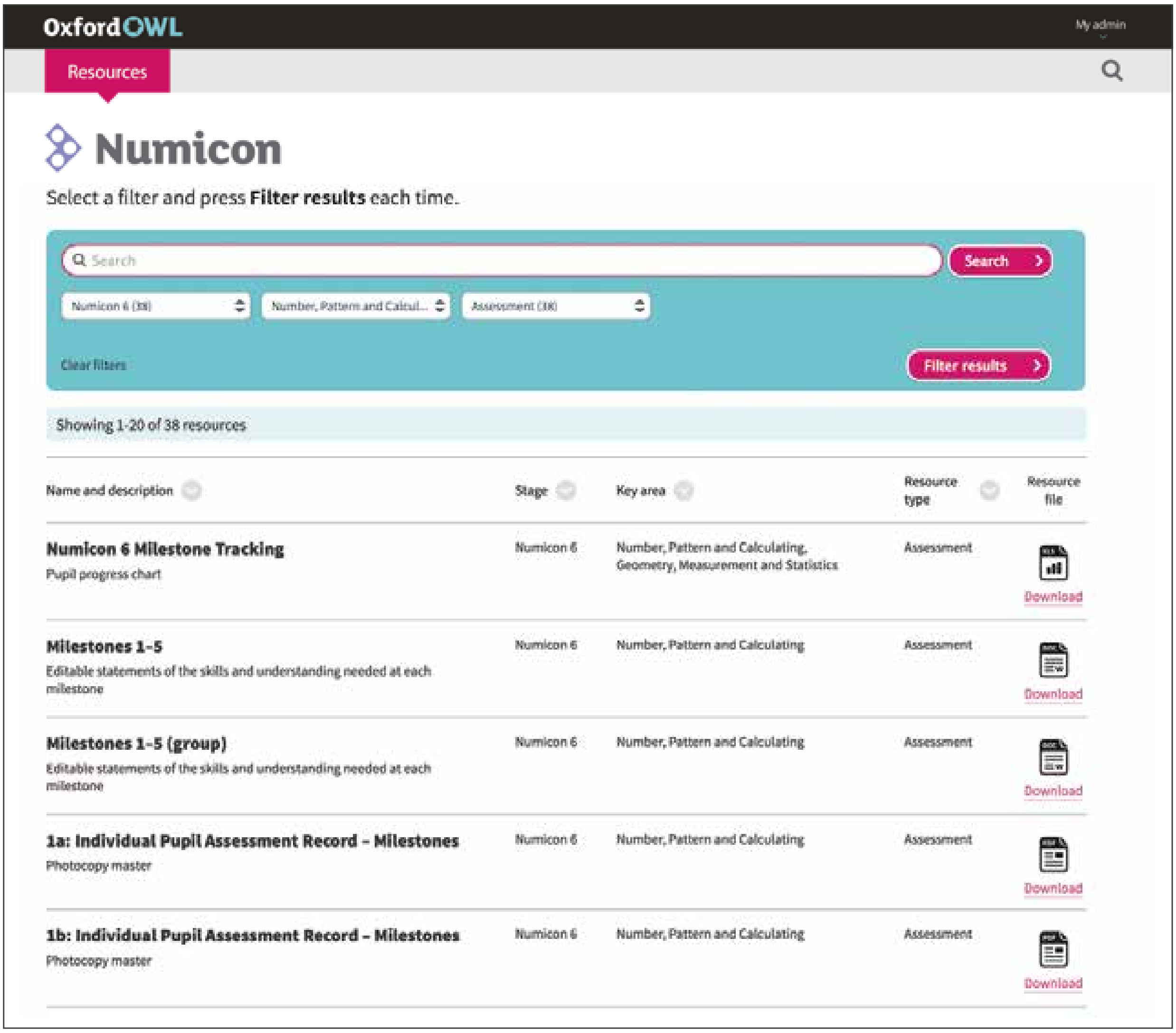
Number, Pattern and Calculating 6 Explore More Copymasters 16

The Explore More Copymasters offer children the chance to practise and discuss mathematics at home with a parent or carer.

An activity has been included for each activity group in the Pattern and Algebra, Numbers and the Number System and Calculating strands, so that children have ongoing opportunities to practise their mathematics learning outside the classroom. There is a bank of additional activities to support the Preparing for Formal Testing strand as a whole.

Each activity is supported by information for the parent or carer on the mathematics that has been learned in class beforehand, and the learning point of the activity itself. Guidance on how to complete the activity is included, as well as suggestions for how to make the activity more challenging or how to develop the activity further in a real-life situation.

The Explore More Copymasters can be given to an adult or child as a photocopied resource.



Numicon Planning and Assessment Support

Additional planning and assessment support for Numicon is available in the Teaching and Assessment Resources on the Oxford Owl website containing a wealth of digital resources that will assist you with your planning and assessment needs.

Within the support, you will find a range of resources including short videos introducing Number, Pattern and Calculating 6 and offering advice that will help you get started teaching with Numicon.

There is also an editable summary of each activity group, including the title and number of the group, the educational context, the learning opportunities, the mathematical words and terms to be used with children as they work on the activities, and the assessment opportunities. There are suggestions for how to use these activity group summaries in your planning, as well as editable short-, medium- and long-term planning frames.

There is a milestone tracking chart to support you in monitoring each child's progress throughout the year. Assessment grids are also available to record children's achievements in each activity group and during work in the Explorer Progress Books.

Charts that map Numicon to the English, Welsh, Scottish and Northern Irish curricula have been included in these resources, as have charts showing the progression of the Numicon teaching programme across Number, Pattern and Calculating and Geometry, Measurement and Statistics.

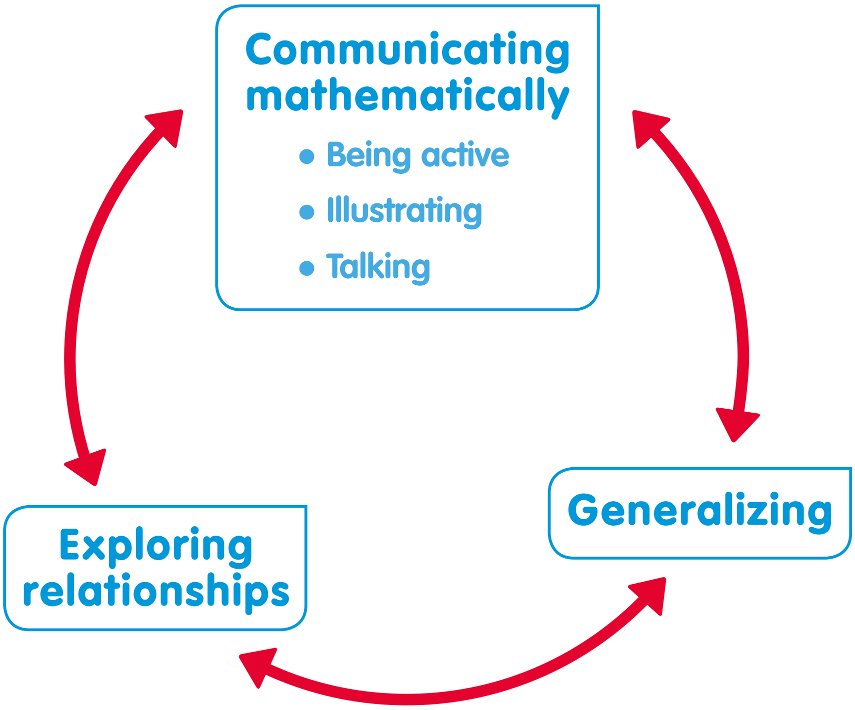
Printable versions of all the photocopy masters are also provided.

What is Numicon?

In order to illustrate how using Numicon in your teaching can help children learn mathematics, this section looks at:

- what Numicon is
- what doing mathematics demands of children
- how using Numicon helps children meet these unique demands.

If you would like further information on the theory behind Numicon from Dr Tony Wing, please turn to page 68.



What is Numicon?

Numicon is a distinctive approach to children's mathematical learning that emphasizes three key aspects of doing mathematics: communicating mathematically, exploring relationships and generalizing.

Communicating mathematically

Doing mathematics involves communicating and thinking mathematically – and these are two sides of the same coin. We think in the same ways that we communicate, and communicate in the same ways that we think. As children learn to communicate mathematically, they learn to think mathematically. This involves them in the following:

Being active: Teaching and learning with Numicon requires children to be active. This does not only mean being physically active (e.g. fitting physical objects together, finding a number on a number line, drawing a shape), but is a requirement reflecting the understanding that mathematics itself is activity: mathematics is something children are learning to *do*.

What this means in practice is that it is always the children themselves who are to do the mathematics. Telling children (or showing, or explaining) 'what to do' can encourage children to be passive. Numicon asks that the children do the mathematics (i.e. both the activity and the thinking). In other words, children actively learn to do mathematics for themselves.

Illustrating: Doing mathematics (i.e. thinking and communicating mathematically) necessarily involves illustrating, because mathematics is about studying relationships between objects, actions and measures, and it is impossible to explore such relationships without some kind of spatial imagery being involved. Even when thinking about just two numbers, one of them 'comes before' the other one on a number line or is said to be 'bigger' or 'higher'; these relations all involve spatial imagery.

Numicon explicitly involves illustrating in every activity. This is how relationships in a wide variety of contexts are both explored and communicated.

Talking: Since doing mathematics involves communicating mathematically (both with others and with ourselves), doing mathematics also involves talking. Talking is an essential aspect of all Numicon activity.

Throughout Numicon, talking means dialogue in which points of view are exchanged between teachers and children, and between children and children. All partners in dialogue will be actively involved, not just passively hearing or waiting to speak. Such exchanges encourage reflective thinking as children learn to discuss different points of view, both with others and with themselves.

Exploring relationships (in a variety of contexts)

Doing mathematics involves **exploring relationships** (i.e. the structure) in any situation in order to develop some kind of control in that area. Relationships can be between amounts, between positions, between shapes, between things that are varying, between things that are constant, or between any combination of all or any of these things. Fully understanding the relationships in a situation makes it predictable, thus making it possible to find out and/or manage what is needed. The mathematical reasoning children are asked to do is their expression of relationships they perceive.

Numicon ensures that children explore the relationships within a wide range of contexts so that they learn not only *how* to do mathematics, but *when* the mathematics they are learning is useful.

Generalizing

In doing mathematics, exploring relationships and looking for patterns in various situations lead to **generalizing**. It is generalizing that allows us to make whole ranges of new situations predictable.

Numbers are generalizations that we all use to make predictions when calculating. For example, the '6', '2' and '8' in the number sentence '6 + 2 = 8' are generalizations; 6 of anything and 2 of anything will together always make 8 things, whatever they are.

'The angles of a triangle add up to 180°' is a generalization that is often used when doing geometry; 'the area of a circle is πr^2' is another that is used when measuring.

In each of these cases, noticing patterns in relationships allows us to generalize about an infinite number of other, similar situations.

It is because we make and use generalizations continually as we do mathematics that mathematical thinking and communicating will appear to have an abstract character for children if they are not involved in doing the generalizing for themselves.

Communicating mathematically, **exploring relationships** and **generalizing** all come together when *doing* mathematics.

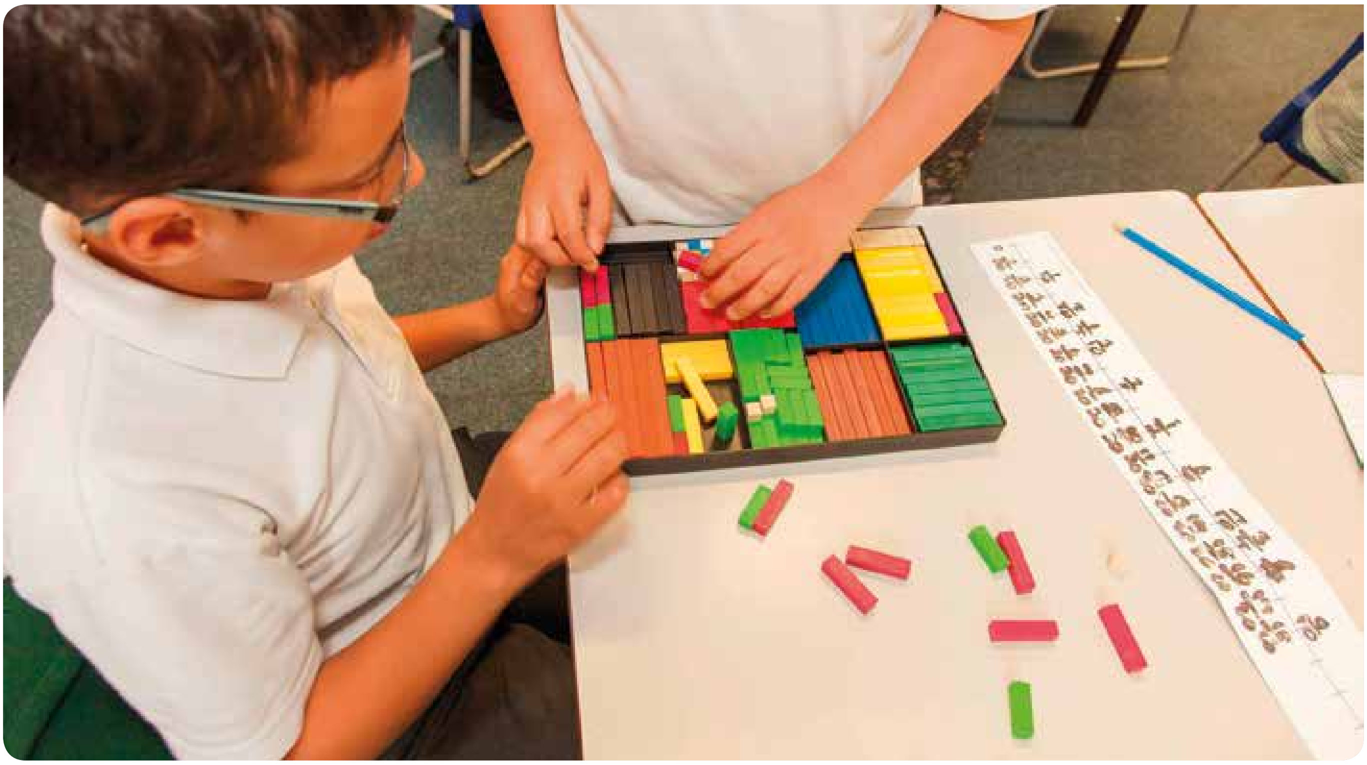
What doing mathematics demands of children

In learning to do mathematics at the age they are asked to, children face a unique challenge in their school curriculum: thinking and communicating about abstract objects.

Pure numbers, such as 6 and 254, are abstract objects and in no other subject are children asked to work with and reason about such things so early. It is not surprising that children can hesitate, be puzzled, get stuck, or take time to work things out for themselves.

Most mathematicians will say that doing mathematics is about looking for patterns in situations. It is, but there is more to it than that. When you find a pattern, it means you have noticed something regular, something that always seems to be happening, and this means that any pattern you see is a generalization of yours. Mathematics deals almost entirely in generalizations.

Which is what numbers are: as generalizations, they are abstract objects. Very early on in mathematics, young children are asked to do things with lots of abstract objects, like those we call '3' and '10'. Not '3 pens', or '10 sweets', or '3 friends'. Just '3' or, even more curiously, the two-digit '10' by itself. Notice how quickly children are asked to add and subtract these abstract objects to and from each other. There are high expectations of every child from the very beginning in mathematics.



The central problem: communicating with, and about, abstract objects

How is it possible to communicate about abstract objects? And, since thinking is communicating with ourselves, how is it possible to *think* about abstract objects?

Notice that ‘abstract’ does not mean ‘imaginary’. We can easily picture imaginary things, such as unicorns and beings from outer space, but abstract things are different. Abstract things include qualities like ‘truth’, ‘intelligence’, ‘common sense’, as well as generalizations such as ‘6 of anything’. The problem is, as soon as you try to picture ‘6 of anything’, you find you are imagining ‘6 of something’.

The answer, as Jerome Bruner observed, is that we think about abstract objects with symbols; in the case of numbers, we do it with numerals.

The important thing about symbols is that they do not attempt to show literally what is being talked about; they are simply arbitrary, conventional marks on a page (or spoken words). When what we are communicating about with these symbols is abstract, that absence of a picture is inevitable; how could we possibly picture something that is abstract?

It is easy to picture ‘3 pens’ or ‘10 friends’, but what might the abstract ‘3’ look like? Or, how about the curious two-digit ‘10’? Since numerals do not look like the abstract things they ‘stand for’, how are children to learn to interpret such arbitrary symbols in their thinking and communicating? Doing mathematics – thinking and communicating about abstract things with symbols – is certainly not easy for young children.

How does Numicon help?

Essentially, Numicon does two things. Firstly, Numicon acknowledges that in order to understand what numbers are, children have to generalize. Secondly, Numicon follows Bruner’s advice in using children’s actions and imagery to prepare for their use of mathematical symbols in their thinking and communicating. In practice, these two things are done at the same time.

In Bruner’s terms, *enactive* and *iconic* representations (action and imagery) are used to inform children’s interpretation of the *symbolic* representation (e.g. numerals) that is necessary for communicating their pattern spotting (generalizing). To aid children’s necessary generalizing, strong focus is placed upon the use of structured materials.

Generalizing and reasoning – an early years example of teaching with Numicon

Initially, in common with most teaching, Numicon involves a wide variety of everyday objects (such as beads, cubes, pegs and counters, see Fig 1) in order to help children develop their counting, before then introducing the challenges of calculating.

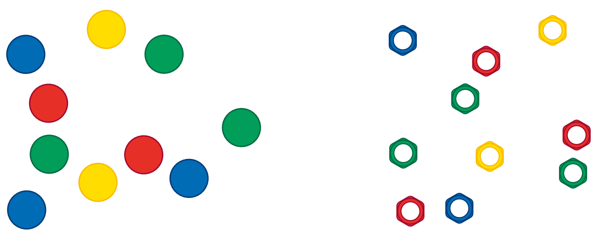


Fig 1

Importantly, Numicon also introduces sets of structured materials in which individual pieces have *regular* physical relationships with each other, for example, Numicon Shapes and number rods (see Fig 2). Children explore the physical relationships within these structured materials by, for example, ordering pieces, comparing them, combining them physically to make others.

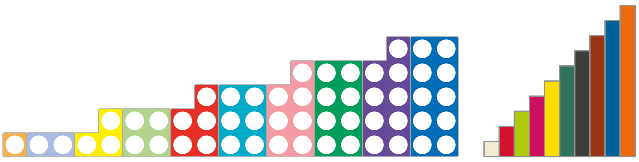


Fig 2

Thus, as children work with loose collections (beads, cubes) and with structured materials (Numicon Shapes and number rods), they are able to combine being active with physical objects and images as illustrations in their talking and thinking about numbers of things.

Numerals are introduced in association with loose collections of objects and with structured materials as children work; thus, symbolic representation appears alongside children's ongoing action and imagery with physical objects.

Number lines (e.g. Fig 3) are introduced to further illustrate the ordering that is evident within the structured materials, and to reinforce the associated ordering of numerals.

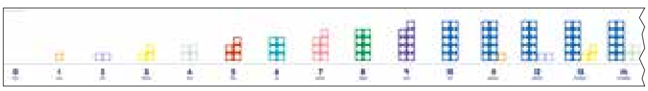


Fig 3

Importantly, loose collections of objects are arranged into the regular patterns of the Numicon Shapes; individual number rods are found to be equivalent in length to different multiples of the smallest 'unit' rod (e.g. Fig 4).

Fig 4

Through doing these activities, children learn that *any* collection of loose objects can be arranged into Numicon Shape patterns that can then be 'read'. Children learn to 'see' how many objects are in a collection without counting; numbers of things begin to 'take shape' visually, in organized ways.

It is now possible to prepare children for further generalizing about numbers through exploring relationships between numbers of things. Children's mathematical thinking and communicating continue to develop through their being active with the objects they are given, and in illustrating their talking about what they see and do.

Children generalize that any collection of loose objects can be arranged into Numicon Shape patterns. They also generalize that any number of 'unit' number rods may be exchanged for (will be 'as long as') one or more of the larger number rods. Thus *any* number of loose objects can be converted into (is equivalent to) one or more Numicon Shapes or number rods.

In realizing that any collection of loose objects can be arranged into Numicon Shape patterns, and that any number of unit

cubes is equivalent to one or more number rods, children become able to 'see the general' in any particular illustration; Numicon Shapes and number rods can be used to illustrate, in an organized way, any numbers of any kinds of things.

Numicon Shapes and number rods themselves may now be used to explore and to communicate about number relationships in general. They have become communication mediators in discussions about numbers and their relationships.

As a particular example, the Numicon Shape that has three holes fits together physically with the Shape that has five holes. The result 'forms the same shape as' (is equivalent to) the Shape that has eight holes, see Fig 5.

Similarly, the number rod worth three units, combined end-to-end with the rod worth five units, are together as long as the rod worth eight units, see Fig 6.

When laid end-to-end along a number line or number track, the '3-rod' and the '5-rod' together reach the position marked '8' on the line.

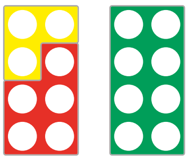


Fig 5

Fig 6

From these actions, and with these illustrations, children are able to generalize that: three *anythings* and five *anythings* together will *always* make eight things.

This generalization can be expressed using the numeric and verbal symbols: 3 and 5 together are equivalent to 8.

Later on, using further actions and further illustrations, children become able to interpret and use the further symbols '+' and '=' to express their generalization thus:

$$3 + 5 = 8$$

Importantly, at this stage children will have begun to use number words (one, two, three) as *nouns* instead of as adjectives (two sweets, three pencils) in their talking.

With their use of number words as nouns, numbers as *abstract objects* have now appeared in children's mathematical thinking and communicating, referred to with *symbols*.

Such generalizing and use of symbols can now be exploited further. If 'three of *anything*' and 'five of *anything*' together always make 'eight *things*', then:

3 tens + 5 tens	=	8 tens
3 hundreds + 5 hundreds	=	8 hundreds
3 millions + 5 millions	=	8 millions

or

30 + 50	=	80
300 + 500	=	800
3 000 000 + 5 000 000	=	8 000 000

Such is the power of generalizations, and of the symbolic notation that children can by this stage use to think and communicate mathematically about them.

Progressing from such early beginnings

The previous example also illustrates how Numicon supports the teaching of children's subsequent mathematics.

In a wide variety of contexts, children are offered opportunities to be active and to illustrate their talking about any (and all) relationships they are exploring between both physical and abstract objects. They do this in order that they may generalize and, as a consequence, think and communicate about their generalizations with the conventional symbols of mathematics: in other words, that they may *do* mathematics.

Of course, there is much more beyond this first example. Children have many other relationships to explore, further generalizing to do, and thus many more conventional symbols to learn to interpret and use. However, wherever it is used, the approach is essentially the same. *Enactive* and *iconic* representation are used to inform children's interpretation of the *symbolic* representation that is necessary for thinking and communicating about their constantly developing pattern spotting (generalizing).

The going gets harder as children progress in their number work: place value, ratios, fractions, decimals and negative numbers will all challenge children's capacity to generalize onwards from these early beginnings. However, such advanced work is equally possible with the same approach of actions and illustrations building towards generalizing, and a consequent use of symbols to think and communicate about the generalized, abstract mathematical objects created. This is how the symbolism of mathematics becomes meaningful.

In their work on geometry, the generalizations that children reach in their mathematical thinking and communicating happen more gradually as they progress towards being able to reason about invented abstract objects such as 'any triangle' and (later) 'any polygon'.

It is impossible to draw the abstract object 'any triangle' in the same way that it is impossible to imagine 'six of anything'. As soon as you draw a triangle, however you have chosen to draw it, you have drawn a particular one; it does not matter whether you draw one that is right-angled, isosceles, equilateral or scalene, what you have drawn is not a general triangle – it is a particular one, e.g. Fig 7.

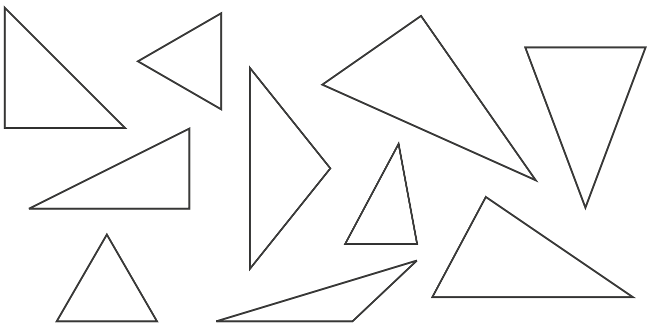


Fig 7

However, as with generalizing about numbers, in doing geometry, much exploring of relationships with action and imagery (enactive and iconic representation) prepares children for reasoning meaningfully about 'any triangle' with symbolic representation (words and symbols).

In this example, as children physically construct and transform many varieties of triangles with dynamic materials, what they see before their eyes is how lengths of sides may vary endlessly as angles change, how angles may vary endlessly as lengths change and yet, despite all the variety they see, the shapes they are creating all turn out to be triangles.

Dynamic mental imagery deriving from these physical experiences then allows children to speak and to think of 'any triangle' as they imagine an infinitely flexible closed 2D shape with three straight sides.

By stressing the straightness of the sides, that there are always exactly three sides and that the sides fit together to 'close' the shapes they make, and by ignoring the constantly changing lengths of sides and sizes of angles, children become able to imagine 'any triangle' and to reason about it with words and symbols.

Then, having generalized to a stage when their mental imagery allows them to imagine and speak meaningfully about 'any triangle', children are in a position to further generalize that, for example, 'the angles of any triangle add up to 180°' through further action, imagery and reasoning with symbols.

In the same way that generalizing about numbers is approached, exploring relationships with action and imagery in geometry prepares children for reasoning meaningfully about abstract mathematical objects with symbols.

We reach all the generalizations of mathematics, and the symbols we use to express them, through being active and illustrating our talking about the relationships we are exploring.

Doing mathematics in the world – solving problems

Of course, *doing* mathematics in the everyday world is not simply about making generalizations and using symbols. Crucially, it also involves making *use* of generalizations to solve problems in particular situations.

For example, the generalization '$4 \times 25 = 100$' allows us to predict that the perimeter of a square of side 25 cm will be 100 cm, that the area of a field measuring 4 m by 25 m will be $100\,m^2$, and that if you save £25 a week for 4 weeks you will have £100. It can also be very useful to help calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

Children need to be able to readily connect the generalizing that they do with mathematical symbols with everyday world situations in which those generalizations are useful.

In their more general problem solving, children learning, for example, *how* to divide one number by another, need to learn *when* that operation is useful.

The Numicon teaching materials organize activities into groups based around mathematical themes. For example, in the *Number, Pattern and Calculating 6 Teaching Resource Handbook*, 'Estimating, rounding and equivalence' (Calculating 3) and 'Using algebra to solve problems' (Pattern and Algebra 3). Each activity group is introduced with, or involves, a context in which that mathematics is useful.

The activities in 'Estimating, rounding and equivalence' involve discussion of appropriate degrees of accuracy and how this varies in different contexts.

'Using algebra to solve problems' introduces the idea of using letters to represent unknown values and explores how this can help in problem solving.

In the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, Measurement 1 children plot distance–time graphs to find average speeds in the context of motorway speed limits.

In these ways, children's mathematics is introduced, wherever possible, within contexts in which that mathematics is useful. Such contexts help children to 'see the point' of the mathematics they are learning, and prevent mathematics becoming a series of answers to problems they have never had.

Flexibility, fluency and persistence

Being able to do mathematics effectively also includes being able to remember basic things such as those generalizations known as 'times tables' and 'number facts'. There are many practice activities built into the Numicon written materials that encourage children to develop a regular familiarity with basic facts.

Even more important to effective functioning is flexibility in mathematical thinking and communicating. Three kinds of flexibility are especially useful.

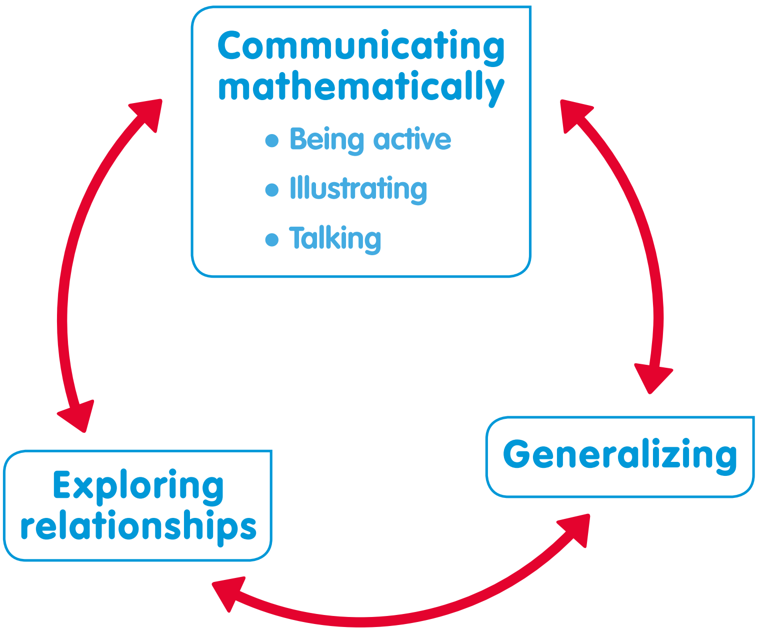
Through being active, children are able to 'invert' their actions; put more simply, this is about 'doing and undoing'. As children have found 'how many' objects there are in a collection by physically grouping them into tens (and hundreds), they are later readily able to 'partition' numbers ('undo' their groupings) when calculating with symbols. As children can combine and separate Numicon Shapes and number rods physically ('do' and 'undo' their actions), they are able to connect adding and subtracting as inverse operations and to check a subtracting calculation by adding.

The introducing and encouraging of a variety of ways of calculating means that children are able to choose methods of calculating that suit the particular numbers involved, rather than adopting a standard method for any calculation. Who would want to subtract 1998 from 4673 using a column method, just because the numbers are big?

As Numicon approaches children's use of mathematical symbols with prior actions and imagery, if children become 'stuck' or hesitant while working with symbols, it is always possible for them to recall and return to the supporting activity and illustration from which their original generalizing arose. For example, if they are working on prime numbers, using symbols, they can immediately return to actively illustrating how factors 'go into' their various multiples physically with Numicon Shapes and with number rods. They can follow a similar pattern with any other kind of number relationship. This flexibility is one of movement backwards and forwards between Bruner's *enactive*, *iconic* and *symbolic* forms of representation in their thinking and communicating.

Finally, because Numicon continuously emphasizes communicating mathematically, it is possible to stress to children that all is not lost when they feel 'stuck'. The thing to do, always, when they 'don't know what to do' in a situation is to communicate.

Being active, illustrating and talking about the relationships we are exploring *is* doing mathematics. Persistence – an invaluable quality when doing mathematics – comes from continuing to communicate, with yourself and/or with others, whenever (for the moment) you cannot see where you are going.



Preparing to teach with Numicon Number, Pattern and Calculating 6

This section is designed to support you with practical suggestions in response to questions about using Numicon in your daily mathematics teaching. It also contains useful suggestions for preparing children for formal national tests that the majority of children will undergo while they are working on Number, Pattern and Calculating 6. There are suggestions on how to plan using the long- and medium-term planning charts, how to assess children's progress using the Numicon materials, and how to make good use of the weeks after the tests.

In this section you will find overviews of:

How does the Numicon approach continue when teaching Number, Pattern and Calculating 6?

The teaching activities in Number, Pattern and Calculating 6 build upon the steady development of mathematical understanding, problem solving, positive attitudes to mathematics and confidence in themselves as mathematicians that children have achieved in their work with Numicon throughout their primary years. While working on Number, Pattern and Calculating 6, the aim is for children to consolidate and maintain their fluency with ideas met previously, and to gain mastery of new mathematical ideas. This will enable them to develop the skills and attitudes necessary to take the artificial conditions of formal national tests confidently in their stride and to be ready for the challenges they will meet in the next stage of their mathematical journey.

National tests will be high in the minds of teachers so it is important at this point to explain that the teaching progression of the activity groups in Number, Pattern and Calculating 6 has been designed to provide a balance between ensuring that children maintain fluency with their previous work, covering new ideas and preparing them for formal testing. Number, Pattern and Calculating 6 includes a new strand, Preparing for Formal Testing, which has been specifically designed to help children develop strategies for fluent arithmetic and intelligent problem solving within the time constraints and pressures of a test environment.

Supporting thinking and communicating with a number rich environment

Numicon classrooms should be rich in number illustration and visually show that mathematics is an important part of children's everyday learning experience. Numicon images and numerals can be incorporated into labels and displays in many areas of the classroom, inviting children's attention. There will be number lines and charts on display where they can easily be seen, including the Numicon 1 000 000 Display Frieze, Numicon 0–1·01 Decimal Number Line, Numicon $^{-}12$–12 Number Line, Numicon 0–100 cm Number Line and a place value chart. A Numicon 0–1001 Number Line may also be displayed.

At different times, displays around the particular aspect of mathematics that children are studying can pose questions and challenges. These can develop into working walls where children post their ideas, questions and solutions that can then be used as a focus for discussion to support and deepen children's understanding. Displays can also celebrate children's work and provide *aide memoires* for systematic approaches to solving problems, number facts and so on. Such displays can change frequently to provide a range of mathematical starting points for enquiry, for example, a picture constructed with composite images, a newspaper cutting involving statistics, an article about record low temperatures.

Although children are likely to sit formal tests without access to structured apparatus or other mathematical materials, working day by day in a mathematically rich environment provides children with valuable opportunities to develop their own mental imagery that will support them in test situations, for example when they may choose to make a quick sketch of some familiar imagery or materials as they work through a problem. Nearly all of us are acutely visually aware, and children are no exception. Throughout the day, and particularly when you are teaching mathematics, you will find the children referring to the imagery and displays as they work on mathematical problems.

Sometimes teachers are concerned that children may become over-dependent on visual structured imagery. However there is no point at which this imagery becomes redundant because meeting new mathematical ideas is ongoing and structured imagery gives children of all ages and abilities a practical, accessible way to investigate mathematical problems by illustrating relationships in ways not provided by written symbols. When children are consolidating an idea, or meeting a new one, structured imagery supports them until they are well on the way to understanding.

Many children, unbidden, move on to working in their heads, using their own mental imagery, while others need more active encouragement to visualize the imagery. A step towards this can be a regular reminder to refer to the mathematical displays around the classroom.

Mathematical communicating in class

The aim of Numicon throughout the primary years is to support children to approach challenges confidently by developing their mastery of the mathematics they have learned and strategies for working systematically through non-routine problems, confident in the knowledge of what they can do when they are stuck. This is achieved through the constant encouragement of children's mathematical communicating. Maintaining this encouragement while children are working on Numicon 6 is crucial to their confidence as they face the new experience of doing maths under test conditions. You will need to encourage children's mathematical communicating constantly and *especially* when the going is tough because their mathematical communicating *is* their 'doing' of maths. Children need to come to understand that it is *communicating* that will always help them when they feel stuck.

The ways in which you communicate and engage in dialogue with children is always a key model for children's own behaviour. By listening carefully to what they say, and responding sensitively with thoughtful questions, you encourage children not only to develop their own thinking and communicating but also to have confidence in their mathematical thinking. It is this confidence that will support them in test conditions and help to prepare them for future challenges in secondary school.

Although children may not be provided with structured imagery in formal test situations, the key role of structured imagery in mathematical communicating and thinking continues in Number, Pattern and Calculating 6 as an essential element within all mathematical thinking.

Seeing a problem modelled with structured imagery can often help to clear children's misunderstandings about the ideas involved, for example in relation to the trickier aspects of multiplicative thinking. Generally encouraging flexibility of imagery will help children to think in different ways about mathematical situations, to communicate more effectively with others and, probably most importantly, to become increasingly capable of trying different approaches to mathematical problems when they feel stuck. See Preparing for Formal Testing 1 on page 196 of the *Number, Pattern and Calculating 6 Teaching Resource Handbook* for an exploration of children using imagery in Numicon 6.

Throughout this year's work children should be encouraged to try different ways of communicating their mathematical ideas and reasoning. This may include making jottings, drawings (children often sketch structured apparatus), formal written methods, Numicon Shapes, number rods, base-ten apparatus, number lines and so on. This also encourages children to think independently as they consider how to communicate their ideas to others, and exposes them to different ways of communicating an idea or an argument with a range of imagery.

Communicating as preparation for formal testing

Children tend to become more adept at organizing their notes systematically as they discover that well organized notes help them to reflect upon and explain their ideas clearly. This systematic noting is an important skill that will help them in test conditions when they are working alone.

As a general rule, children should be encouraged more and more during their final year at primary school to 'think for themselves' about mathematical situations, and consequently feel able to try several ways literally of 'looking at' a mathematical activity. Having confidence in their own thinking and being willing to tackle new ideas will allow children to trial many approaches to problem solving when faced with challenging test questions (see 'How do I deal with children who are stuck?' on page 31).

Importantly, the more that thoughtful dialogue is encouraged daily within a classroom, the more children become able to think for themselves. Through working with others to explore mathematical situations widely, and constantly looking at situations from another person's point of view, most children get used to the idea of looking at situations in different ways, and doing so becomes a habit – a habit that will support them enormously in a test situation.

Getting started with the Number, Pattern and Calculating 6 teaching programme

The plan–teach–review cycle applies to Numicon, just as it applies to all effective mathematics teaching. There are, however, three important features of Numicon that support this cycle.

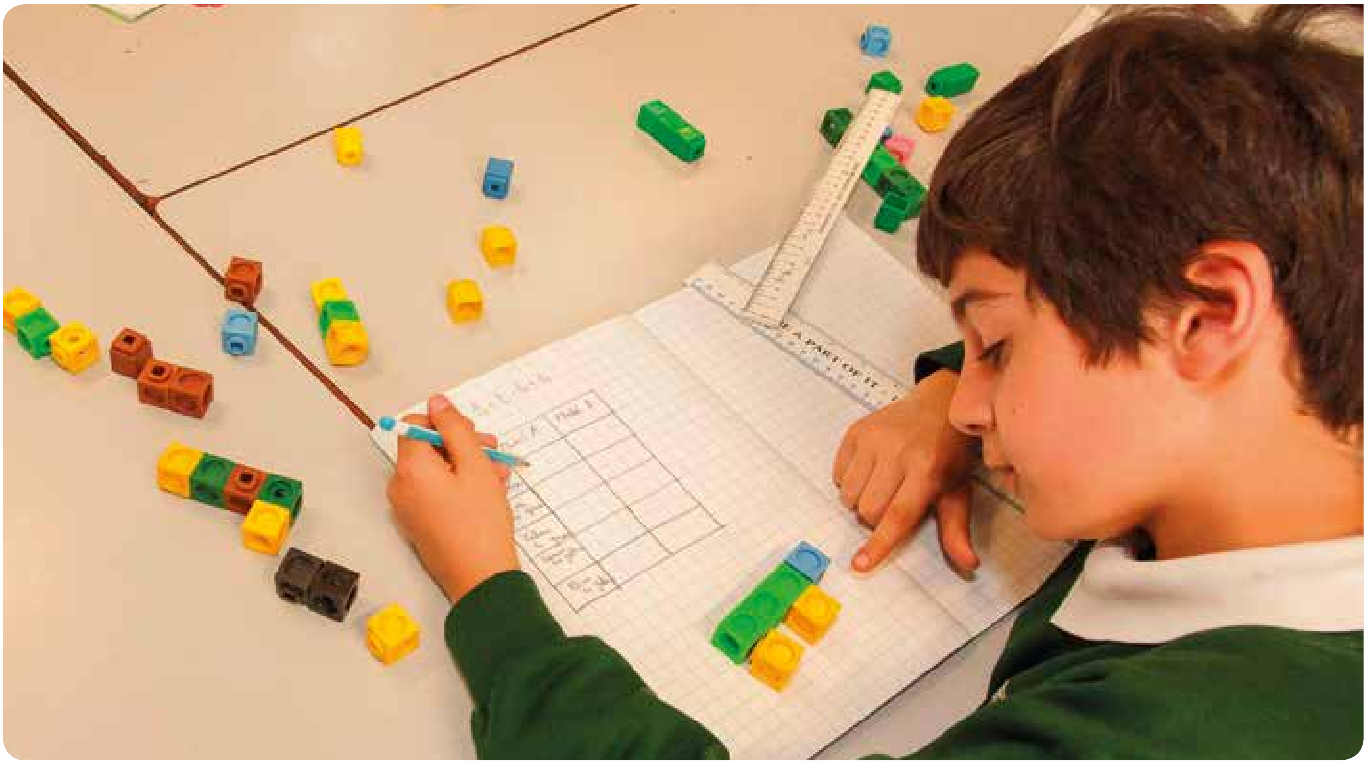
Firstly, the Numicon teaching programme for Number, Pattern and Calculating is arranged into three core strands: Pattern and Algebra, Numbers and the Number System and Calculating. Within each of these strands is a sequence of activity groups, though the strands are interrelated and what children learn in one strand supports their learning in another.

The long-term plan found on page 18 of the *Number, Pattern and Calculating 6 Teaching Resource Handbook,* shows the recommended order for teaching the activity groups (also available as an editable version on the Number, Pattern and Calculating 6 Teaching and Assessment Resources on Oxford Owl). This plan has been carefully designed to scaffold children's understanding so that they are able to meet the challenges of each new idea, e.g. children will not be expected to explore prime factorization until they are very familiar with the underlying ideas of factors and multiples.

Practice and discussion activities are included within each activity group, some for the whole class and others for individual or paired work.

Finally, what many people call 'using and applying' does not need to be planned separately, although children will need plenty of practice in non-routine problem solving. All the activity groups start with a problem to be solved and the cumulative nature of the teaching programme means that children are *using* their earlier learning every time they face a new idea or situation.

The medium-term planning guide on page 20 of the *Number, Pattern and Calculating 6 Teaching Resource Handbook,* gives expected coverage for each half term and lists the activities and the learning opportunities for each activity group. Time is precious in the months leading up to formal



tests so you may decide to follow the long- and medium-term plans as they stand but timetable extra sessions to cover the material thoroughly. You may need to plan some further practice activities to maintain children's fluency with arithmetic techniques and methods they have learned previously.

There are summary charts showing the title and learning opportunities for each activity group in the long- and medium-term planning charts in the *Number, Pattern and Calculating 6 Teaching Resource Handbook* and available on Oxford Owl.

The key to the activity groups on pages 34–35 of this Implementation Guide (also included in the *Number, Pattern and Calculating 6 Teaching Resource Handbook*) highlights the key parts of each activity group.

The majority of the activity groups begin with a relatively low threshold focus activity, to include all children and support confidence. The remaining focus activities are designed to help children progressively develop their ideas around the theme of the activity group. You will notice that there are opportunities for reasoning about numbers throughout focus activities through some quite challenging questions.

The focus activities may be used for whole-class and group teaching; some may be taught quite quickly to the whole class and explored later with a group. Children who quickly accommodate the ideas they are meeting may do more than one focus activity in a single focus teaching session in order to deepen their experience.

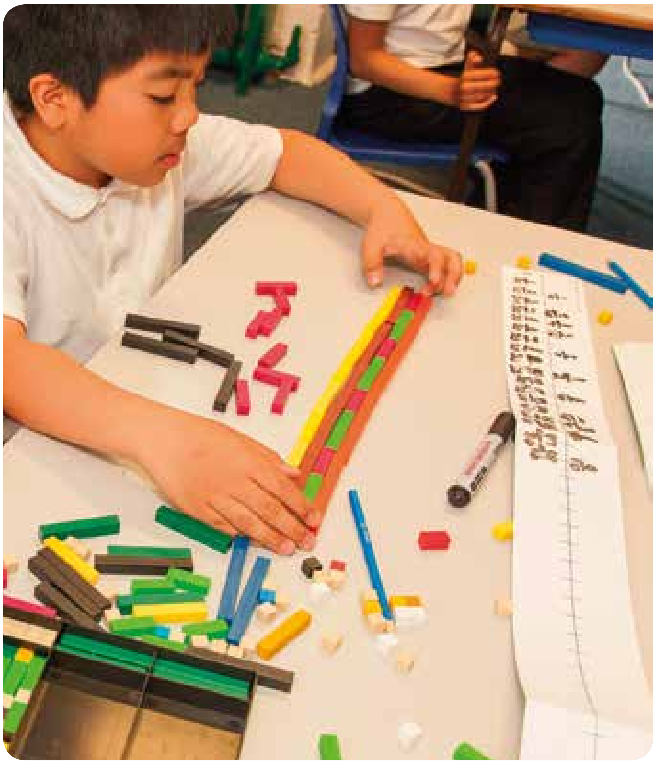
Some elements to include in mathematics lessons:

- Support children's confidence by ensuring that there is a balance between activities that will challenge children and those that children are likely to do more speedily.
- Since Numicon activities are all 'themed', you may decide to allocate different activities to different groups of children.
- Remember to check that there is scope for all children to take their activity further.
- You can increase challenge through your questions to specific groups and by asking challenging questions when the class comes together for the final part of the lesson.
- For morning maths meetings, build in whole-class practice activities from earlier activity groups to help children develop and maintain fluency.
- Build in practice constantly with both mental and formal written methods of calculating.

How long should I allow for teaching each activity group?

The Numicon teaching activities for Number, Pattern and Calculating 6 primarily address the number and algebra areas of the mathematics curriculum. Time can feel short at the end of primary school, with new material to be taught and revision for national tests, so each activity group provides sufficient material for at least one week with practice activities that can be revisited over time.

It is highly improbable that you would expect all the children to do all the activities in every activity group but the detailed progression and range of focus activities is there to provide flexibility for teachers to exercise their professional judgement as to which children need to work carefully through the earlier activities in a group and those who may move on quickly to the later, more challenging activities. If a child is having difficulties getting to grips with an aspect



of the mathematics you are teaching, refer to the detailed progression order of the long- and medium-term planning charts in the Teaching Resource Handbook to find earlier coverage of the topic in Number, Pattern and Calculating 6, or in earlier years.

As you teach the activities you will find that some children will move on very quickly and you will be able to combine two or sometimes even three activities within one teaching session. These children may well complete all the activities in the group but careful questioning is advised in order to check that understanding is sufficiently deep.

Other children may need longer to establish secure understanding but will cover several of the activities and benefit from returning to finish the activity group after a week or so. This has the advantage of reminding children about ideas they have met earlier and gives you a useful opportunity to review what they have remembered. For further assistance, refer to the medium-term planning chart in the *Number, Pattern and Calculating 6 Teaching Resource Handbook,* page 20, and consider when might be the best time to ask children to work on the relevant Explorer Progress Book pages.

When considering which activities to utilize from an activity group, refer to the relevant milestone statements from the medium-term planning to guide you in your selection of material for your children. Milestones pick out those aspects of mathematics that are absolutely essential for children's progress, and so specify what they cannot afford to miss.

The majority of children embarking on Number, Pattern and Calculating 6 will have been taught with Numicon in the preceding years and will be familiar with the approach's emphasis on communicating. Refer to previous tracking records of children's progress against the assessment milestones in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* for useful evidence of which children have achieved mastery of the ideas they have learned and are ready to engage confidently with the new ideas they will meet in Number, Pattern and Calculating 6.

For children who have not worked with Numicon at all before, refer to the *Number, Pattern and Calculating 5 Teaching Resource Handbook*. The first activity group, Getting Started, provides a series of activities focused on how to use structured apparatus to represent mathematical operations and number ideas. This has two main benefits: children have to carefully reflect on their mathematical understanding as they consider how to use structured apparatus to communicate their ideas, and their actions with the materials often give clues as to their level of understanding for teachers.

At the beginning of the year

The first activity group (Preparing for Formal Testing 1) focuses on children's self-assessment to help teachers and children identify any areas they feel they most need to work on to establish sufficient understanding for the Number, Pattern and Calculating 6 activities. The activity itself simply involves putting a selection of challenges into four categories: 'I think I could do these easily', 'I think I would probably get these right', 'I think I could try to get these right', and 'I know I would struggle to do these'. Note that children are not asked to do the mathematical tasks, simply to judge their perceived difficulty. This self-assessment activity can also be repeated at intervals to show progress, including a few weeks before the national tests to identify areas for some last minute revision. Children are usually quite accurate at judging what they find difficult.

Children's responses to the self-assessment activities in the first activity group will have indicated where practice and revision is most needed. It may be beneficial to some children to revisit activity groups in Number, Pattern and Calculating 5 to secure understanding and then to practise frequently to establish fluency.

The second activity group (Preparing for Formal Testing 2) focuses on systematic approaches to non-routine problem solving; this is deliberately included at the beginning of the year as children will need plenty of regular and collective experience of this approach to problem solving from the start of the year. There is a corresponding spread at the start of Explorer Progress Book 6a which contains some problems for children to help ascertain their level of understanding of the approach. There is a second spread for this activity group in Explorer Progress Book 6c which offers more problems for children to work on in the weeks before the national tests. They should get used to thinking about their own thinking, so they will be ready to use the approach independently in a formal test environment.

Continuing the teaching programme

The subsequent nineteen activity groups address new material for Pattern and Algebra, Numbers and the Number System and Calculating to be taught before children sit national tests. They all include opportunities for systematic problem solving and children should have regular practice at solving multi-step non-routine problems involving all four operations. Children are encouraged to develop their repertoire of fluent calculating methods, including formal written methods, which they can use flexibly in response to different situations, including checking the accuracy of an initial calculation by repeating it using a different method.

Preparation for formal testing

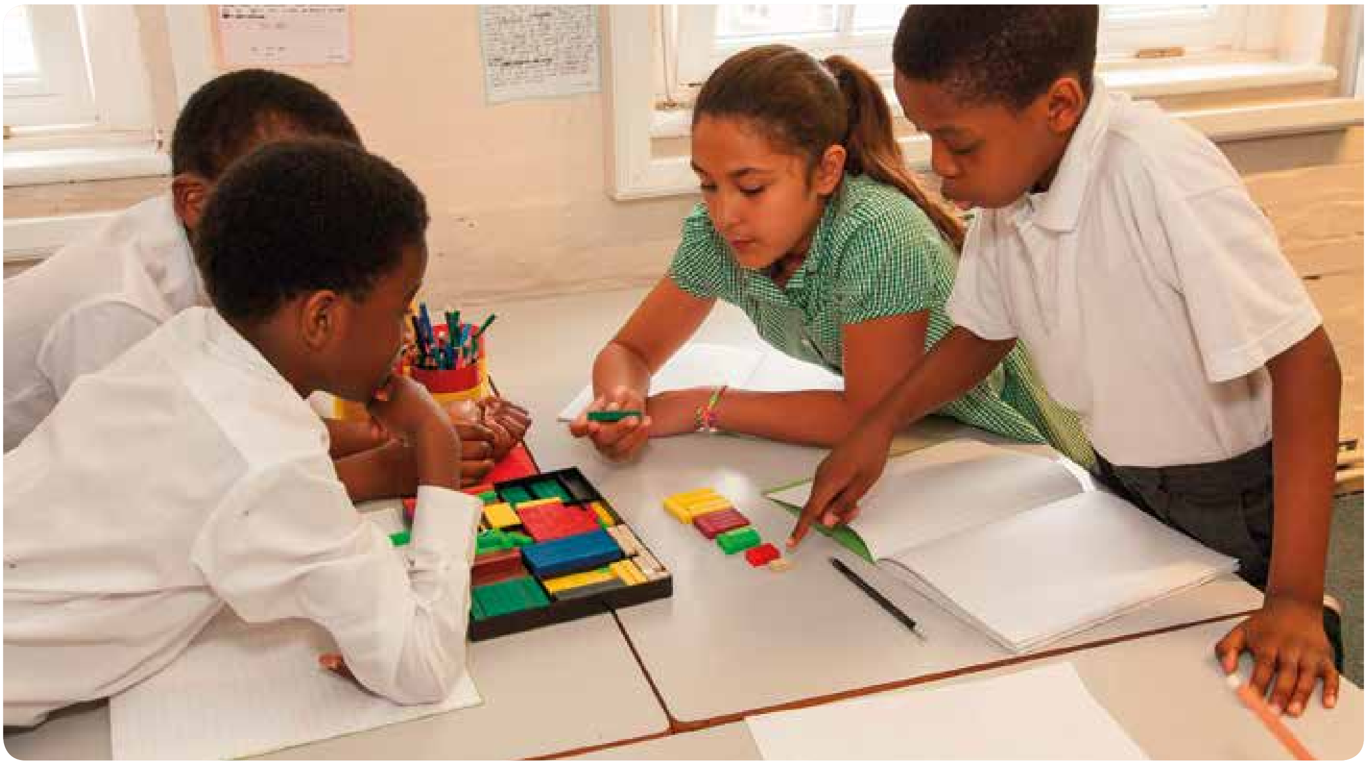
So that children go into test situations confidently, knowing what to expect and how to deal with the challenges they will encounter, there are three Preparing for Formal Testing activity groups that are designed to be taught during the final weeks before children sit national tests. These activity groups are designed specifically to prepare children for the experience of doing mathematics in a formal test situation where they will be working alone, in silence, on questions normally posed in formal written mathematical language.

There are two types of national test required to be taken by children of this age in England: the arithmetic test in which children are asked to complete a number of routine calculations correctly, and mathematical reasoning papers in which children are asked to think through a number of non-routine problem situations, sometimes set within an everyday context, sometimes within purely mathematical contexts, e.g. a problem involving prime numbers.

Preparing for the arithmetic test involves remembering methods, such as column addition and subtraction, and long multiplication and division, and appreciating the need to make a realistic estimate of the answer and to know how to check the answer. Children also need to reason how to work through a calculation when they cannot remember the method. Preparing for Formal Testing 3 and 4 (Fluency in calculating with whole numbers and decimals and Fluency in calculating with fractions and decimals) are designed for children to rehearse calculating techniques, use number relationships, and generally to increase their fluency in calculating.

The reasoning tests present children with non-routine tasks. Faced with these children may well feel stuck and not know how to proceed, and in the test situation they have to rely on their own resources. Problem solving strategies introduced at the beginning of Number, Pattern and Calculating 6, and used regularly throughout the year, will have helped to prepare children for facing non-routine situations alone.

The final test preparation in Preparing for Formal Testing 5 (Preparing to do maths in test conditions) is designed to help children do their best in both the arithmetic and reasoning papers. It provides a series of activities and topics for discussion across three broad areas of preparation: reading questions carefully, using time most effectively, and what to do when you feel stuck and alone. This helps children to feel emotionally prepared for the test situation, to avoid misreading questions, to give themselves time to think, to pace themselves appropriately and to respond thoughtfully and positively if they feel stuck during a test (see 'How do I deal with children who are stuck?' on page 31).



Investigating

There are six Investigating activity groups for the final term of Number, Pattern and Calculating 6, offering genuinely useful opportunities for children to explore aspects of the mathematics they have mastered in the preceding months and years. Children investigating in mathematics are essentially following their own thoughts, decisions and questions around a particular topic. Typically, investigating begins with an initial question that either occurs to a child spontaneously, or has been given to a child as a particular 'starting point'.

Since investigating is by its nature child-led, it can only progress under conditions in which children are quite free to make their own decisions and evaluate their own thoughts and questions. Importantly though, children are not left to flounder on their own while investigating, but are encouraged to share their thinking as it develops, through discussion and debate with others. Thus children may investigate either individually, or in small groups, but will always have the opportunity to discuss and share their current progress, puzzles and findings with others.

While investigating, children are invited to think mathematically in an open situation. This will involve **being systematic** (exploring possibilities systematically, in an organized way), **generalizing** (looking for patterns, working out a general rule and predicting), and **being logical** (reasoning, to predict, to confirm and to explain).

The role of a teacher while children are investigating is to set a personal example. The 'openness' of investigating experience will necessarily involve children feeling at times lost, stuck, and indecisive, as well as by turns eager, angry, excited, depressed, thoughtful, curious, proud and bemused. Teachers, through their own example, should help children accept that such feelings are entirely natural and normal in open and novel situations, and that in spite of feeling lost, stuck, and even sometimes completely without hope, there is almost always something that can be done to improve and develop a situation.

The major benefit children gain from investigating lies in their becoming personally more resilient, more resourceful, and more trusting of their own questions and thinking. Children who learn to trust their own thinking will be much more persistent when tackling new problems and new topics at other times in their mathematical careers. By giving children time and opportunity to follow their own thinking and questioning while they investigate, we show them that we value the quality of their individual thinking.

Very importantly, while investigating children depend a great deal upon using their own powers of reasoning. In their dialogue with others they will have maximum opportunity to convince others about the validity of their particular findings, and also to question the reasoning and findings of others. This is essentially how mathematical reasoning develops – in dialogue with others.

Time limits

Because investigating is an open-ended activity, there is no knowing in advance how long it may last. In practice it is useful to set a maximum time limit, say one week (or perhaps two), by the end of which children will be expected to have prepared a presentation of their findings in relation

to a particular starting point. Children who decide they have finished earlier than this can prepare their presentation and then begin to work from another starting point.

Content limits

Because children are following their own questions, there is no knowing in advance where these may take them. In practice however, given starting points tend to focus thinking in particular areas, and teachers will have opportunities to guide children with carefully chosen 'What if…?' questions, such as 'What if those numbers were…?', while discussing progress with a child. Importantly, there are no specified end-points to any investigating, no pre-determined 'learning objectives' in relation to any mathematical content expected to be covered.

Intervening

Teachers intervene simply with conversation while children are investigating, inviting children to explain what they have been doing, and discussing possible future moves. Informal and sensitive questioning during discussion can guide children when they seem stuck, and encourage their persistence. A particularly useful guide to helpful questioning can be found in the introductory section of the *National Numeracy Strategy Mathematical Vocabulary Booklet* (2000).[1] Teachers are not expected to have the answers to any of a child's questions; their role is to encourage exploration and, by their personal example, to demonstrate a willingness to investigate openly.

1 Available in the National Archives: http://webarchive.nationalarchives.gov.uk/20110202093118/http:/nationalstrategies.standards.dcsf.gov.uk/node/84996

Writing for production

Since investigating is in its very nature experimental, children who are investigating need to have maximum freedom to write, draw, cross out, sketch, note, scrawl, and generally make repeated revisions of their initial work. This means that while investigating children need plenty of rough paper and materials on which to 'sketch out their thoughts', to try out illustrations, arguments, and generally to experiment – visually, and in highly personalized note forms.

Writing for presentation

A vital part of any investigating is the presentation to others of one's findings, usually generalizations of some kind together with supporting reasoning that both justifies and explains the validity of the conclusions. For example, a child might conclude that there are only six possible ways of making or doing some particular thing, and the accompanying reasoning will demonstrate 'why'.

Obviously, writing for presentation has to be clear to others and so is quite different to the informal personalized note forms and sketching that happen during investigating itself. Children can work towards their final presentation of reasoning in stages: first 'convince yourself' that what you are saying is true, next 'convince a friend', and finally 'convince an enemy', that is, someone who is trying to find legitimate ways of disagreeing with you at every step. Presenting should involve the most demanding mathematical dialogue, as conclusions are questioned as thoroughly as possible.

Working with mixed abilities

Clearly in open situations some children will appear to 'achieve more' than others. Hence in order to help *all* children become more resilient, resourceful, and trusting of their own thinking and questioning it is important to celebrate a wide range of qualities and outcomes to investigating. Being persistent, being systematic, and being thorough should be celebrated as much as being imaginative, going further, and being articulate.

Ultimately, all children should be supported to enjoy the freedoms of truly open exploring, of being encouraged simply 'to ask whatever questions you want' of mathematics.

Organizing, managing and resourcing Numicon maths lessons

Timetabling

The depth and reach of the mathematics that children are meeting in Number, Pattern and Calculating 6, and the subsequent extended nature of some of the activities may mean breaking away from the fixed self-contained daily maths lesson to allow sufficient time for teaching and learning more difficult ideas and to prepare for formal tests. Consequently flexibility in the timetable is important and the format of maths lessons may vary.

Sometimes there may be a whole-class introduction, often illustrated with imagery through the use of the *Numicon Software for the Interactive Whiteboard*, followed by a longer session of group work during which children are either working independently or being taught in a focused group, and ending with the class coming together for a concluding session. Alternatively children may continue to work on a problem or practice activities begun previously.

The emphasis on mathematical communicating should be maintained throughout all activities, however. At different times children may be working in a small adult-led group, participating in a class conversation, conversing with a partner or engaging in a small group discussion sharing ways of finding solutions to problems, illustrating their ideas with different imagery, but having a common theme to the activities of an activity group allows for continuing rich classroom mathematical dialogue, shared by all.

Grouping

By the time children are working on Number, Pattern and Calculating 6, some will have achieved mastery of the maths learned in previous years and will be able to engage confidently with new ideas and move on to activities that will deepen and broaden their understanding of these ideas. Others may need more time to consolidate their understanding of ideas met earlier and extra practice to develop fluent recall of key number facts and calculating procedures. These differences need to be taken into account when planning lesson content and organizing the class.

Some schools group children of like ability. Although there will still be a range of needs within classes, grouping children who are ready to work on similar challenges can economize on teaching time. However, it is important to be aware that there are risks involved in ability grouping that can have negative impacts, for instance by putting an artificial ceiling on expectations of children's ability and achievement; by

creating an air of complacency in higher ability groups; by children in lower ability groups seeing themselves as failing in maths. Children placed in either a high or low ability group irrespective of their ability are likely to take on characteristics of that group; misplacement can therefore result in able children underachieving.

Mixed ability classes provide opportunities to be flexible about grouping so children can sometimes work in mixed-ability groups and at other times with children working at a similar level. Whatever the policy is in your school, bear in mind that it is important to vary groups from time to time to ensure that children do not always work with the same partner.

Best practice suggests that, if available, the teaching assistant works with all children at different times, as does the teacher. This helps to keep expectation of achievement high and reduces the likelihood of children becoming over reliant on adult support. To encourage communicating, children who are working independently on practice activities usually work either individually or in pairs within a larger group.

Resourcing practical activities

Equipment suggested for each activity is listed in the '**Have ready**' section. (Note that photocopy masters can be printed from the Number, Pattern and Calculating 6 Teaching and Assessment Resources on Oxford Owl *or* photocopied from the *Number, Pattern and Calculating 6 Teaching Resource Handbook*.) Sometimes, when you are teaching a new idea, you may decide to put out the equipment before the lesson to save time. However, as the majority of children will be used to working on practical activities using apparatus, they could instead collect the necessary equipment for themselves. To facilitate this it is important that resources are clearly labelled and easily accessible.

Encouraging communicating with a daily 'maths meeting'

Holding a daily oral and practical 'morning maths meeting' lasting about 15 minutes has proven to be very successful in encouraging children's mathematical communicating. Such meetings include a small selection of key routines every day in which children practise rapid recall to maintain fluency with calculating methods and number facts learned previously and to develop their communicating with ideas they are meeting currently. For suggested activities refer to whole class practice activities in the activity groups; you might also discuss observations about a mathematically rich object or picture that you or a child has brought in.

At other times work on solving non-routine mathematical problems as a class, perhaps one that has come up in the class, in the school, in the news, at home, or one selected from the activity groups. Children's reasoning and recall can also be encouraged throughout the day. For instance, if a couple of children are needed to help with something in the school you could select them by asking questions that will require them to think carefully, e.g. 'is your birth date a multiple of 7?' or 'is your birth date a prime number?'.

Communicating practically and on paper

Throughout the Numicon teaching programme, there is an emphasis on *doing*, often with practical apparatus and discussion before children try working with their ideas on paper. However, by the time they are working on Number, Pattern and Calculating 6 children will be writing down their ideas more frequently. Giving children choice about

how to communicate their ideas on paper can provide useful insights into how they approach problems, how comfortable they are with formal written methods, whether they are working systematically and how they are using conventional notation.

Developing and maintaining fluency

The activity groups provide sufficient material to allow for a whole class, low threshold introduction, followed by a range of focus activities for teaching the key ideas involved. Children who understand the new mathematics quickly may often assimilate more than one focus activity during a lesson and move on to use the independent practice activities, which are largely open-ended, to develop fluency and deepen their understanding. Allow more teaching time for children who work more slowly and may need teaching in smaller steps to help them grasp the ideas. Children who advance more quickly with an activity should be encouraged to explain their thinking and approaches to the whole class.

At the end of each activity group is a list of suggestions for whole-class and independent practice and discussion activities to help children develop fluent understanding of the ideas they are learning. After the teaching has moved on, help children develop fluency, understanding and accuracy in their calculating from previous activity groups by ensuring that they revisit and apply their reasoning and methods regularly in as wide a variety of contexts as possible. These opportunities might arise within other activity groups in Number, Pattern and Calculating or Geometry, Measurement and Statistics 6, and in other curriculum areas.

Other opportunities for mathematical thinking might occur incidentally, e.g. what percentage of the day do we spend in the classroom? What is the average age of the class? What percentage of the class is left handed? What is the difference in average temperatures between the last two weeks? Current events in the school also provide opportunities, e.g. calculating how much each class has contributed to a sponsored event (calculating the total and rounding it to the nearest pound, then working out what percentage of the total each class contributed), working out how many tickets each family might be allocated for the school performance and a fair way of allocating any remaining seats.

These sorts of non-routine problems expose children to a wide range of different contexts in which mathematics is useful, and provides useful practice in the mathematics children are learning as they work out what they need to use to solve each problem.

There are also Explore More Copymasters for Number, Pattern and Calculating 6, which provide further opportunities for children to practise and discuss at home the mathematics they have been working on at school.

How do I deal with children who are stuck?

It is important that children know there is nothing wrong with difficulty and that they are not afraid of it. They need to know they are not expected to realize straight away how to start working on a non-routine problem; solving it will perhaps mean that they feel stuck to start with and will need to try different ways of approaching the problem. Throughout their work with Numicon children are encouraged to communicate mathematically when challenged and to persist through difficulty by self-reflecting and explaining what the difficulty seems to be, using illustrations and actions to express their

problems. Encouraging children to express and deal with difficulty will also help them to respond positively if they are stuck during a test, when they will need to explain the problem to themselves silently and visualize any relevant mental imagery.

The activity group Preparing for Formal Testing 2 on non-routine problem solving should be first used near the beginning of the year. It is designed to build on earlier work to prepare children for non-routine problem solving in test situations by introducing a three-phase approach to problem solving:

- entry – children work out what the problem is about and the challenges it presents, use illustrations and think about how they will go about solving it;
- attack – children decide where to start, what to do next and what to do after that and follow their plan through;
- review – children check that their solution is reasonable, sensible and in the correct units and if necessary check any calculating in other ways.[1]

Give children plenty of practice of using this approach to non-routine problem solving and encourage discussion about their experiences to help them build confidence in their ability to persist when the going seems tough.

Preparing for teaching from an activity group

Understanding the mathematics yourself

Before you teach an activity group, read the relevant sections from the 'Key mathematical ideas' chapter. Ask your own questions. Don't assume you 'already know this'; we all forget things and need to remind ourselves of the details. If there are things you don't fully understand, consult other sources, such as Derek Haylock's *Mathematics Explained for Primary Teachers* (5th ed., 2014).

Remember, mathematical thinking and communicating essentially involves much generalizing, so always look for, and be clear about, the *generalizing* to be done in any activity group. In Number, Pattern and Calculating 6, children are invited to generalize both formally and informally. For example, in Pattern and Algebra 2 children are asked explicitly to describe the nth (general) term of a sequence formally, as a formula, whereas in Calculating 5 children might notice informally that percentages are usually much easier to compare than common fractions.

Always ask yourself what the patterns and regularities are in the work that children need to look for, and try to be prepared for both children's formal and their informal observations.

The children may be new to an area, so ask yourself, 'What patterns in the work they're doing will children have to notice in order to progress?' Another way of working on the generalizations involved is to keep asking yourself and the children 'Will that always work?', 'What if those numbers were different?', 'Would that work with fractions?', 'Will that ever work?', '*When* does that work?', and 'What never works?'

Now you know what the mathematics in an activity group is about, consider, 'When is that mathematics *useful*?'

1 See Mason, J., Burton, L. & Stacey, K. (2010) *Thinking Mathematically*. New Jersey: Prentice Hall

Appreciating the contexts

The educational context on the introductory page for each activity group will help you to see how the ideas are developed and how they fit into the continuum of children's learning about Number, Pattern and Calculating.

Think about the kinds of contexts offered in the activity group you're working on; is this mathematics useful in particular kinds of 'real world' situations, or will it help me do some other mathematics? It can be helpful to think of one or two contexts of your own, just to be sure that you understand what the *point* of doing this mathematics is.

Children don't just need to know *how* to do this mathematics, they need to know *when* to do it, as well. How can you help them spot when this *general* mathematics applies to a *particular* situation?

Understanding the illustrating and communicating involved

Whatever mathematics the children are doing, if they are using numbers then they are working with generalizations *they cannot see*. So what illustrations are available to you and the children? In what ways might illustrations help children 'to see the general in the particular'? For instance, if you want children to generalize that 'it doesn't matter which way round you add two numbers, the total will always be the same', then using visual Numicon Shapes or number rods as illustrations is very immediate.

And which symbols and words are key to successful communicating in this area? Are any of these new? How are children going to *join in* with the ways these terms are normally used when doing mathematics? Each activity group suggests key terms; the idea is that through doing the activities the children will learn how such terms are used. Encourage them to use them in their conversations and let them hear how you use these terms.

Review all the activities of the activity group in terms of the mathematical communicating involved. What actions, imagery, and conversations are children going to be using and having during this work?

Selecting and adapting activities

First of all, check the learning opportunities and the educational context of the activity group, which will give you an overview of the ideas children will be meeting, and the previous learning that it builds on. Check that children have achieved mastery of these previous ideas. Each activity group starts with an introductory low threshold activity that is designed to be accessible to all children in the class. Children's responses will help you select which activities from the group you will give priority to as the work develops; refer also to the relevant milestone for a reminder as to what is considered essential.

Read *all* of the activities in the activity group. Consider what each activity contributes to the overall work of the activity group. Then try these activities for yourself.

You know the children in your class, and the materials available to you. Provided *you* can see the point of each of the activities, you will be able to select and adapt creatively. Some activities might be 'revision', others might involve ideas, illustrations or techniques that are completely new. Some activities will be invaluable practice. If your earlier assessments indicate that some children are not yet ready for the level of challenge in the activities, think about how you might adapt these and if necessary look back through earlier activity groups in the same strand, even in previous years, to find appropriate activities and work from these to establish children's understanding. Another activity might need to be adapted to increase the challenge. Be flexible: adapt what is available for your children in the light of what they can already do.

You are likely to plan to cover more than one focus activity when teaching those children who readily assimilate the new ideas. They are likely to move on quickly to the more challenging activities later in the activity group. The open-ended nature of the activities and the emphasis on mathematical thinking will provide scope for these children to take activities further. You may decide to increase the challenge further through planning specific questions.

Encourage children to remember the point about 'difficulty': difficulty is *normal*. When children get stuck, their job is to *communicate*. Make sure they have continuously available all the materials and imagery they might need to communicate their difficulty effectively.

Using the activity groups

The first page of each activity group is clearly coloured according to the **strand** it appears in (Pattern and Algebra – red, Numbers and the Number System – yellow, Calculating – dark blue, Preparing for Formal Testing – dark red, NPC Investigating – light purple). The title and the number of the activity group allow you easily to identify the content and how far through the strand you are.

The **key mathematical ideas** clearly highlight the important ideas children will be meeting within each activity group.

The **assessment opportunities** signal key information to 'look and listen' for, which indicates how much of the focus activities children have understood.

The **educational context** gives a clear outline of the content of the activity group as well as, e.g. how it builds on children's prior learning, how it relates to other activity groups and the foundation it establishes for children's future learning.

All activity groups have been extensively trialled in the classroom, so the **learning opportunities** come from real classroom experiences. They are designed to help children develop their understanding of the key ideas of an activity group.

Key mathematical ideas Generalizing, Pattern, Algebra, Functions, Inverse, Equivalence, Mathematical thinking and reasoning

Pattern and Algebra

Using symbols and letters for variables and unknowns 4

Educational context

In this activity group, children continue to explore how to describe general situations and rules mathematically. They are supported to express patterns numerically, e.g. as sequences and functions, and to identify and describe relationships between numbers, e.g. as formulae. This links to children's work in the *Geometry, Measurement and Statistics 6 Teaching Resource Handbook*, Measurement 2 (Area of 2D shapes). This leads into describing general rules which apply in any instance of the same type of situation, and, building on their work in Pattern and Algebra 3, to expressing these rules concisely using algebra, with letters standing for unknown values and variables. For example, in Activity 6 they work out how to describe the commutative property of adding two numbers – the property that the order in which the numbers are added doesn't matter – more succinctly, as $a + b = b + a$.

Connecting with the work of Pattern and Algebra 2, we explore general rules of divisibility for help in finding factors.

Learning opportunities

- To describe a numerical pattern or general relationship in words and algebraically, as a formula.
- To recall and use tests of divisibility by 2, 3, 5, 9 and 10.
- To describe and explain the commutative property of adding and multiplying.

Words and terms for use in conversation

algebra, algebraic notation, symbol, generalize, reasoning, logic, systematic, show, prove, pattern, sequence, constant difference, term, first term, term-to-term rule, predict, relationship, general rule, general term, nth term, unknown, variable, value, expression, equation, equivalent, inverse, function, function machine, input, output, divisibility, test of divisibility, factor, multiple, prime, composite, commutative property, associative property, number trio, part–whole relationship

Assessment opportunities

Look and listen for children who:

- Use the words and terms for use in conversation effectively.
- Can identify the term-to-term rule in a linear sequence, e.g. in the sequence 38, 43, 48, 53, … the term-to-term rule is 'add 5'.
- Describe a rule for finding the general term of a linear sequence and express this with an algebraic expression, e.g. $5n + 33$ in Activity 1.
- Can explain algebraically how 'think of a number' problems work.
- Can explain the general relationship between an 'input' (x) and an 'output' (y) for a particular function (e.g. for a function described by $y = 3x$, y is always three times x, x is always one third of y).
- Can identify a missing input or output for a given function machine, and a missing instruction, e.g. '× 3' for a given set of inputs and outputs.
- Can write an equation to show the general relationship between input and output for a given function, represented as x and y respectively, e.g. $y = 3x$.
- Use tests of divisibility to sort numbers.
- Describe the commutative properties of adding and of multiplying in general terms, including algebraically, e.g. $a + b = b + a$, $ab = ba$.
- Can explain why adding and multiplying are commutative, while subtracting and dividing are not.

Explorer Progress Book 6b, pp. 20–23

After completing work on this activity group, give small focus groups of children their Explorer Progress Books and ask them to work through the challenges on the pages. As children complete the pages, assess what progress they are making with the central ideas from the activity group. Refer to the assessment opportunities for assistance. Children will also have the opportunity to complete their Learning Log (pp. 22–23) where they can reflect on the mathematics they have done so far.

Explore More Copymaster 4: Secret Function Machine

After completing work on Activity 4, give children Explore More Copymaster 4: Secret Function Machine to take home.

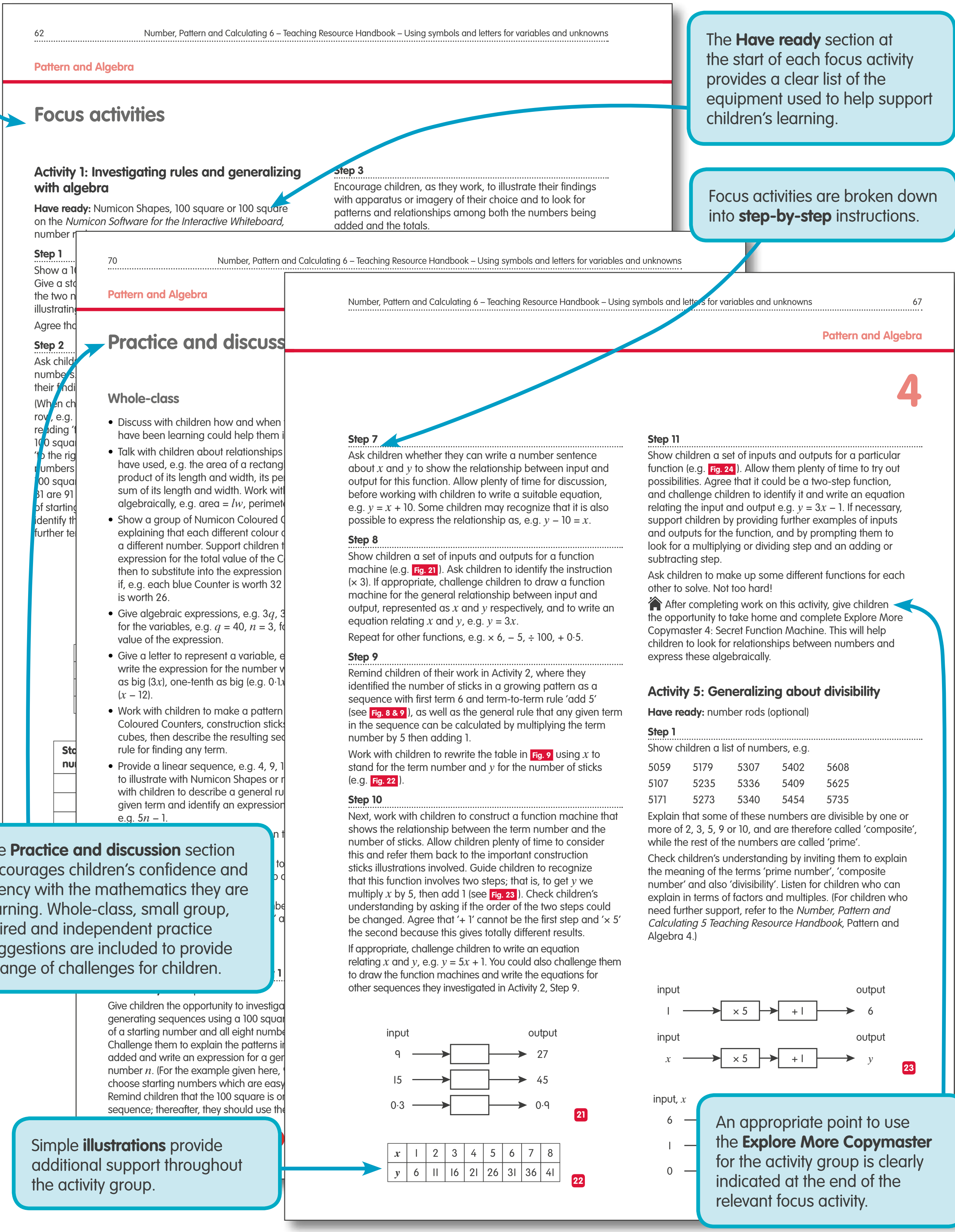
Explore More Copymasters provide an opportunity for children to practise the mathematics from the activity group outside the classroom through fun, engaging activities.

Clear links are made to the **Explorer Progress Books**. These books provide an invaluable chance to see children's thinking, monitor their progress and assess how much of the activity group they have understood.

Each activity group includes several **focus activities**, each clearly titled to show the specific learning points addressed. The first focus activity is a relatively 'low threshold' activity allowing all children to engage with the work. Focus activities then build progressively to a 'high ceiling' that provides challenge and allows for some differentiation within the same activity group, where necessary.

The **Have ready** section at the start of each focus activity provides a clear list of the equipment used to help support children's learning.

Focus activities are broken down into **step-by-step** instructions.


62 Number, Pattern and Calculating 6 – Teaching Resource Handbook – Using symbols and letters for variables and unknowns

Pattern and Algebra

Focus activities

Activity 1: Investigating rules and generalizing with algebra

Have ready: Numicon Shapes, 100 square or 100 square on the *Numicon Software for the Interactive Whiteboard*,

Step 3

Encourage children, as they work, to illustrate their findings with apparatus or imagery of their choice and to look for patterns and relationships among both the numbers being added and the totals.

70 Number, Pattern and Calculating 6 – Teaching Resource Handbook – Using symbols and letters for variables and unknowns

Pattern and Algebra

Practice and discuss

Whole-class

Number, Pattern and Calculating 6 – Teaching Resource Handbook – Using symbols and letters for variables and unknowns 67

Pattern and Algebra

4

Step 7

Ask children whether they can write a number sentence about x and y to show the relationship between input and output for this function. Allow plenty of time for discussion, before working with children to write a suitable equation, e.g. $y = x + 10$. Some children may recognize that it is also possible to express the relationship as, e.g. $y - 10 = x$.

Step 8

Show children a set of inputs and outputs for a function machine (e.g. Fig. 21). Ask children to identify the instruction (× 3). If appropriate, challenge children to draw a function machine for the general relationship between input and output, represented as x and y respectively, and to write an equation relating x and y, e.g. $y = 3x$.

Repeat for other functions, e.g. × 6, – 5, ÷ 100, + 0·5.

Step 9

Remind children of their work in Activity 2, where they identified the number of sticks in a growing pattern as a sequence with first term 6 and term-to-term rule 'add 5' (see Fig. 8 & 9), as well as the general rule that any given term in the sequence can be calculated by multiplying the term number by 5 then adding 1.

Work with children to rewrite the table in Fig. 9 using x to stand for the term number and y for the number of sticks (e.g. Fig. 22).

Step 10

Next, work with children to construct a function machine that shows the relationship between the term number and the number of sticks. Allow children plenty of time to consider this and refer them back to the important construction sticks illustrations involved. Guide children to recognize that this function involves two steps; that is, to get y we multiply x by 5, then add 1 (see Fig. 23). Check children's understanding by asking if the order of the two steps could be changed. Agree that '+ 1' cannot be the first step and '× 5' the second because this gives totally different results.

If appropriate, challenge children to write an equation relating x and y, e.g. $y = 5x + 1$. You could also challenge them to draw the function machines and write the equations for other sequences they investigated in Activity 2, Step 9.

x	1	2	3	4	5	6	7	8
y	6	11	16	21	26	31	36	41

22

Step 11

Show children a set of inputs and outputs for a particular function (e.g. Fig. 24). Allow them plenty of time to try out possibilities. Agree that it could be a two-step function, and challenge children to identify it and write an equation relating the input and output e.g. $y = 3x - 1$. If necessary, support children by providing further examples of inputs and outputs for the function, and by prompting them to look for a multiplying or dividing step and an adding or subtracting step.

Ask children to make up some different functions for each other to solve. Not too hard!

After completing work on this activity, give children the opportunity to take home and complete Explore More Copymaster 4: Secret Function Machine. This will help children to look for relationships between numbers and express these algebraically.

Activity 5: Generalizing about divisibility

Have ready: number rods (optional)

Step 1

Show children a list of numbers, e.g.

5059	5179	5307	5402	5608
5107	5235	5336	5409	5625
5171	5273	5340	5454	5735

Explain that some of these numbers are divisible by one or more of 2, 3, 5, 9 or 10, and are therefore called 'composite', while the rest of the numbers are called 'prime'.

Check children's understanding by inviting them to explain the meaning of the terms 'prime number', 'composite number' and also 'divisibility'. Listen for children who can explain in terms of factors and multiples. (For children who need further support, refer to the *Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 4.)

The **Practice and discussion** section encourages children's confidence and fluency with the mathematics they are learning. Whole-class, small group, paired and independent practice suggestions are included to provide a range of challenges for children.

Simple **illustrations** provide additional support throughout the activity group.

An appropriate point to use the **Explore More Copymaster** for the activity group is clearly indicated at the end of the relevant focus activity.

Planning and assessment cycle

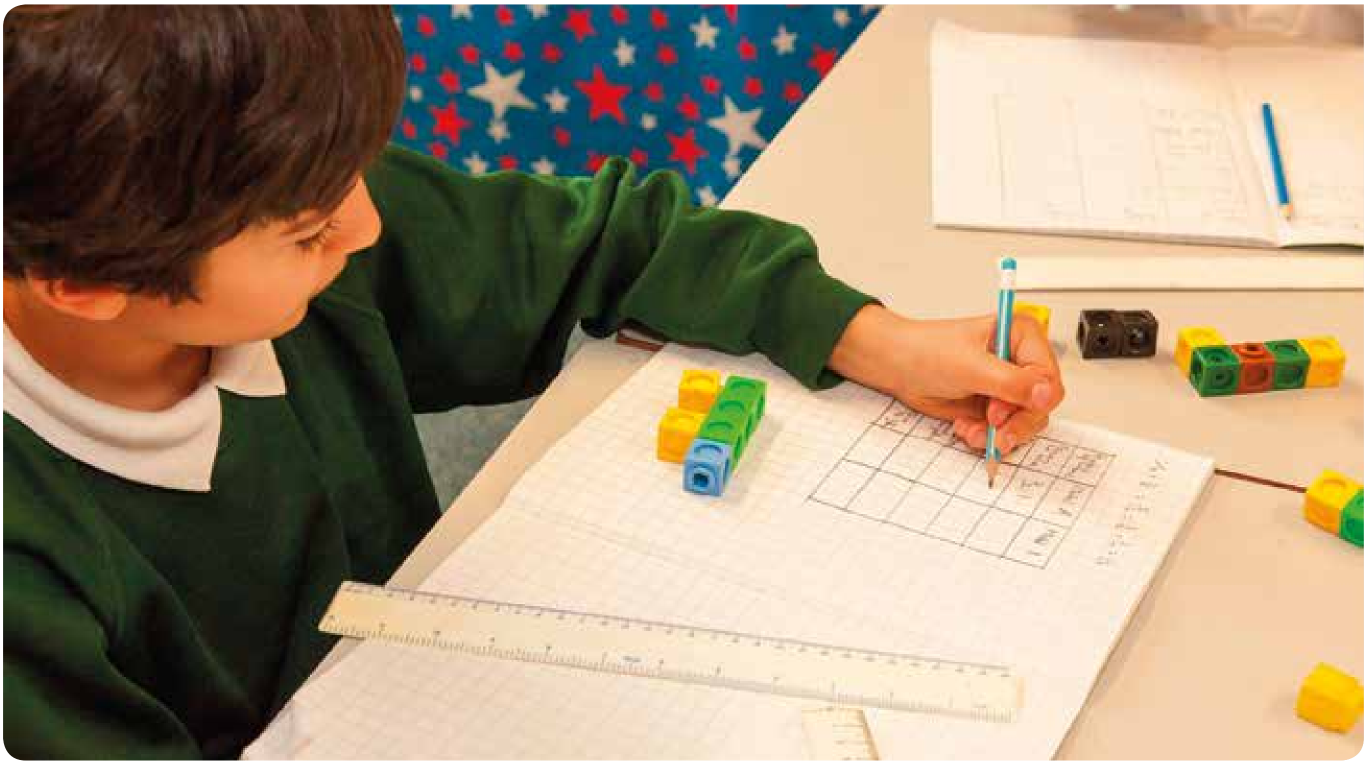
Here is a guide to show how planning can be informed by your assessments of children's understanding.

1. Choose an activity group	Before embarking on the activity group, review the educational context, consider your initial assessments of children and decide whether there is a need to revise any ideas children have met earlier.
2. Choose a focus activity	If this is the first lesson using the activity group, start with an early 'low-threshold' activity to include all children. Before the lesson, check the learning and assessment opportunities, brief teaching assistants and prepare resources.
3. Choose the practice activities	**Independent groups:** Refer to your assessment notes and allocate suitable practice activities (found at the end of the activity group). **Focus teaching groups:** Refer to your assessment notes and the learning and assessment opportunities from the activity group and allocate a focus activity.
4. Plenary session (normally during and at the end of lessons)	Think about the important ideas that children will have met in the lesson, particularly any generalizations that you want children to have made. Plan questions to prompt discussion and ask questions that encourage children to reflect on ideas they may have learned. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
5. After the lesson	Reflect on how children have responded in the lesson and note any significant steps on your assessment records. Use what you notice to determine the plan for the next lesson. Suggestions are given for whole-class practice that will help children to develop the ideas they have learned in the lesson. At some point after children have completed work on the activity group, ask them to complete the relevant pages of their Explorer Progress Book. This will allow you to assess how well they have retained the information they have been learning. It will also give you a chance to see how well they are able to apply this knowledge when faced with a 'non-routine' problem.

Creating short-term plans

Here is a template for how you might create a short-term plan. An editable version of this template can be found in the Planning and Assessment Support.

	Warm-up	Main Teaching Focus	Focused Group Work with the Class Teacher or Teaching Assistant	Independent Work	Plenary
Activity number/title	Select activities from the **Whole-class practice** section. This could be from a previous activity group to review and refresh children's previous learning.	Select one of the focus activities from the activity group matched to the needs of the children. Place the **activity number/title** of the chosen focus activity in your short-term plan.	Decide whether to: • select the next **activity number/ title** from the focus activities in the activity group. Place this in your short-term plan or: • consolidate the activity covered in the main teaching focus.	Decide whether to: • choose activities from the **Independent practice** section for groups, pairs or individual children. Make notes on your plan or work from the Teaching Resource Handbook or: • select a focus activity for groups to work on independently. Place the relevant **activity number/ title** in your short-term plan.	Encourage children to have a reflective conversation to draw together what has been learned in the lesson. Refer to the end of the activity group to find suggestions for some whole-class practice questions.
Learning opportunities	Place the selected learning opportunity(ies) from the chosen activity group summary in your short-term plan.				
Notes and Educational context	Decide whether to: • use the activity directly from your *Number, Pattern and Calculating 6 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 6 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	Decide whether to: • use the focus activity from your *Number, Pattern and Calculating 6 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity. If working with a teaching assistant, you may want to select the relevant **educational context** from the chosen activity group.	Decide whether to: • use the practice or focus activity from your *Number, Pattern and Calculating 6 Teaching Resource Handbook* or: • draw on the Teaching Resource Handbook to make your own notes for teaching the focus activity.	
Words and terms	Decide which **words and terms** you will use in **conversation**. Place these in your short-term plan.				
Resources	Prepare any resources you may need for the activity. Use the **have ready** section at the beginning of the focus and practice activities.				
Assessment opportunities	Select from the chosen activity group the **assessment opportunities** that you and the teaching assistant will be looking and listening for in the different parts of the lesson. Place these in your short-term plan. Remember to note whether children know when to use their understanding.				



Assessing

Encouraging children's reflection and self-assessment

Whatever the format of the maths lesson, ask questions of individuals, groups or the class about the maths they are working on to remind children to continue to self-assess by reflecting on their learning. As children are working, very often someone will put forward an idea that is worth everybody considering. In this case, you might choose either to invite all children to take a moment to reflect on the idea, or make a note to discuss this at the end of the lesson.

You may also have points you want to draw to children's attention, e.g. a strategy you have noticed a child using which you want to draw to the attention of the class. To help children's more general reflection at the conclusion of their lesson, you may ask them to think quietly for a few moments about what they have been doing and guide them with questions such as:

- How did you approach the problem?
- Did you make an estimate? Was this close to your answer?
- Was there anything you could have done differently?
- Could you find ways of checking your solutions?
- Is there anything new you have learned?
- Is there anything you feel particularly pleased about?
- Is there anything you particularly enjoyed?
- Is there something that you found difficult or that you did not understand?
- Is there something that you are still puzzling about?
- Is there something that you would like to do again?

Assessing children's progress for teaching

Assessing children's mathematics using Numicon involves making judgements about developments in children's mathematical communicating – both receptive and expressive.

Each activity group lists several assessment opportunities that point to key achievements to look for during work on that activity group. Familiarize yourself with these before you begin your teaching on any activity group to help guide your interactions with children as they tackle the activities you give them. Children's achievements will be evident in their actions, imagery and conversation as they progress. Specifically, you will need to look for developments in children's actions (what they do and notice), the imagery they use and respond to, and their use of (and responses to) words and symbols in their conversation. Your assessments will provide useful information for planning your teaching as you move on to the remaining activity groups.

Remember also to notice children's fluency; when is their communicating stilted, full of gaps and hesitations, and when does it flow consistently and well, suggesting a strong command of connections between well-established ideas? This ongoing formative assessment should be as open as possible so that children can communicate as much as possible. It is through their mathematical communicating that you will gain real insight into how children are thinking. This will enable you to make the most accurate assessment of their progress.

If it helps, list the assessment opportunities on a sheet of paper against a list of the names of children you are teaching, or print the Assessment Grids for Number, Pattern and Calculating 6 from Oxford Owl, so that it is easy to quickly note individual achievements and difficulties as their work progresses. As well as the assessment opportunities for the whole activity group, within each activity, there are also suggestions for what to 'look and listen for' as children are working on the activities.

Focus on children's communicating and ask whether they know *how* to do the mathematics they are learning, and whether they know *when* to use it. How they communicate will give you an insight into their understanding. For instance, a child who is working by trial and error with muddled explanations probably does not yet understand the activity. Plan to revisit it, focusing carefully on the mathematical language and imagery you will use. It may be that the child did not understand what they had to do. If children are self-correcting, that is, working by trial and improvement, this suggests that their understanding is developing as they try out different solutions. Give children time to experiment and practise the activity and encourage discussion about their ideas. Children communicating clearly about what they have done (by talking, with apparatus, and by writing it down), suggests solid understanding. If you notice some children are always the first to finish, or their books show only correct answers, then perhaps the challenge is insufficient for them: ask more searching questions of their work.

Assessment milestones and tracking children's progress

Within the medium-term plan for *Number, Pattern and Calculating 6 Teaching Resource Handbook*, you will notice that there are milestones (summary statements) of what children need to be close to mastering before they move on to the next section of activity groups.

The statements in each milestone are founded on the assessment opportunities in the preceding activity groups and are also aligned to the 2014 National Curriculum.

Your ongoing assessment of each child will build up over the preceding period and you can keep a record of each child's attainment and track their progress using the individual photocopy master of the collated milestones for the year (Number, Pattern and Calculating 6 Teaching Resource Handbook, photocopy masters 1a–1b, or Individual Pupil Assessment Record – Milestones on Oxford Owl). You could also track pupil progress using the editable Numicon 6 Milestone Tracking spreadsheet, available on Oxford Owl.

At each milestone you can reflect on each child's achievements and decide whether you need to plan further support and practice for them, giving them time to consolidate their understanding, or whether they are ready to move on. If children are moved on before they are ready, then their difficulties are likely to accumulate because they will not be adequately prepared for the new ideas they will meet.

Explorer Progress Books

These offer specific challenges for the purposes of assessing. Children cannot pass or fail at these assessment tasks, they simply respond in their individual ways. How they approach the tasks and their detailed responses will demonstrate their mathematical thinking and communicating and give you an opportunity to 'see' their thinking through the illustrating they use within the tasks. This level of insight into children's thinking will make it easier to gather meaningful and accurate assessment of where children are. The Assessment Grids for Number, Pattern and Calculating 6, available as editable files on Oxford Owl, contain suggestions of what to look for in children's work and allow you to track their achievements. Preparing for formal test situations is different and is addressed through the specific test preparation activity groups (see also Preparation for formal testing, page 26).

Key mathematical ideas

Underlying the activities in Number, Pattern and Calculating 6 are many key mathematical ideas that children will be developing and extending, as well as some symbolic conventions they may be meeting for the first time.

In order to teach these ideas and conventions effectively, those who are working on activities with children will need to be very clear themselves about the mathematical content involved in each activity.

The following section includes a brief outline of the key mathematical ideas that children will be encountering in the activity groups of the Number, Pattern and Calculating 6 Teaching Resource Handbook. Discussion of the mathematical ideas within this section will help with planning how best to develop these ideas with children.

The educational context page of the activity groups lists the key mathematical ideas associated with each activity. As you prepare for your teaching, you may find it helpful to remind yourself about the key ideas behind each activity by referring to this section.

The mathematics coordinator may also find it useful to work on these key mathematical ideas in professional development sessions with class teachers and the wider school staff.

In this section you will find overviews of:

Fluency, reasoning and problem solving – the aims of the 'Mathematics programmes of study: key stages 1 and 2 National curriculum in England 2014'

Mathematics is an activity, and 'doing mathematics' essentially involves communicating and thinking in distinctive ways. Numicon activities are devoted to supporting children's development of their mathematical thinking and communicating – both with others and with themselves – through inviting them continually to be active, to illustrate and to talk.

Fluency is an attribute of communicating to do with an effective smoothness of flow; it should not be confused with being able to do something quickly. Understanding is involved in fluency, and a simple mechanical calculating proficiency alone is not enough for children to become fluent in doing mathematics. Knowing how to add doesn't help much if you don't also know *when* to add.

Numicon activities develop richness in children's thinking and communicating by encouraging them to be active, to illustrate their thinking and to talk. Such richness in communicating helps children to respond flexibly to challenging situations, and encourages them to develop the habit of always looking for a variety of ways of communicating about (and hence thinking about) the structures and possibilities of new situations. This flexibility of thinking and communicating supports mathematical fluency in new and unfamiliar situations.

'Automaticity' supports fluency as children develop and progress to facing ever more complex situations. As every learner driver understands very well, many actions and responses need to become automatic in order to drive a vehicle smoothly and effectively in complex traffic situations. So it is with doing mathematics; the more automatic certain responses become, the more able we are to direct available attention to reflecting and acting on the more complex aspects of problem situations. With Numicon, the practice necessary to developing familiarity and automaticity is integral to all activity groups.

Flexibility and automaticity together are what allow children quickly to assess any calculation as either 'One that I could do this way', or as 'One I could change into something much easier'. In all the open discussion encouraged throughout Numicon activities, in work on 'non-computational thinking' and in the explicit emphasis upon algebraic relations that underpin effective calculating, children are encouraged to approach calculating in a thoughtfully fluent manner, rather than mechanically. This is fluency based upon understanding, not upon a blind remembering of little-understood rules.

Reasoning in mathematics is about much more than simply using words and symbols logically. Imagery is involved in almost all mathematical thinking and reasoning, and so is action. For example, reasoning about sequences of 'square' or 'triangular' numbers, or even about some numbers being 'bigger' than others, clearly involves action with imagery.

Even though all that might appear on a child's page are written numerals and other symbols, the communicating the child has done or is doing with themselves (that is, their thinking) is likely to also involve action, imagery and words. Reasoning about abstract number ideas depends upon visual imagery such as number lines and patterns to communicate the 'logic' of the interrelationships involved.

Following a line of enquiry, conjecturing relationships and generalizations, and developing an argument, justification or proof using the full range of mathematical communication are integral to mathematics teaching with Numicon.

Problem solving – addressing a mathematical challenge that the solver doesn't immediately know how to approach – will involve both reasoning and thinking flexibly as well as requiring imagination, courage, and persistence. Number, Pattern and Calculating 6 aims to help children develop these qualities in four key ways.

Firstly, to support children's approaches to problem solving, the mathematics they are learning is grounded within contexts in which it is seen to be useful. If children can 'see the point' of the mathematics they are being asked to learn, they are a good way towards knowing when else that mathematics would be useful.

Secondly, there is an acknowledgement that doing mathematics is usually challenging – unless children are in a familiar situation that they have recognized. It is expected that children will find most Numicon activities non-routine and suitably challenging, and that challenge will therefore become normalized. When children are challenged – and especially when they are 'stuck' – it helps them enormously to try to communicate the difficulty they are experiencing as fully and richly as possible.

This is where the rich variety of communicating that is continually encouraged with Numicon activities comes into its own; as children try to communicate a difficulty in as many ways as they can, new ways forward will almost always occur to them. The central message to children is that when they don't know what to do, they should aim to communicate mathematically and express their difficulty as precisely as possible.

Thirdly, in Number, Pattern and Calculating 6 all of children's earlier work on place value, calculating, estimating and approximating bears fruit as they tackle more and more problem situations realistically. In everyday life and work we rarely meet problem situations that are neat and tidy, or that address just one kind of calculating alone; at this stage in children's mathematics their judgement is increasingly required in deciding which calculations, and what levels of accuracy are appropriate.

Finally, in the Explorer Progress Books, children are regularly supplied with unfamiliar and quite open situations that invite them to use the mathematics they have been learning. These experiences are designed both to confirm the message that challenge is normal, and to identify particular aspects of their work that will benefit from further attention.

By facing mathematical challenge, and by developing their resilience and resourcefulness in the face of such challenge, children are learning to solve both routine and non-routine problems as independently as possible. Problem solving can then become a familiar and enjoyable experience.

Communicating and thinking mathematically

Children learn to think mathematically by learning to join in with the ways we, as practised mathematical thinkers, communicate mathematically. Their thinking is in fact their own developing version of the mathematical communicating they meet around them, both within classrooms and in the wider world.

Mathematical communicating and thinking is distinctive in several ways; mathematics is essentially about looking for patterns and regularities in situations, generalizing about them, and thus gaining control within a context, whether it be about shapes, quantities, weather forecasting or relationships of any kind. Mathematical communicating and thinking typically display the following key features:

1. **Generalizing**: In doing mathematics we are constantly generalizing, and then generalizing about our generalizations. In our early years we might generalize that every time we add five objects to four objects; whatever they are, we always end up with nine objects altogether, and we learn to communicate this generalization conventionally in mathematics by writing and saying:

$$4 + 5 = 9$$

Later on, as we get more used to working with numbers, we start to make new generalizations about these early generalizations, such as: 'Whichever way round you add two numbers, they always come to the same total.' We would write and say this conventionally in mathematics as:

$$a + b = b + a$$

This new equation[1] uses letters because we want to make a general statement about adding *any* two numbers, and we cannot say or think this by using particular numbers – it wouldn't be a general statement if we did. We use 'a's and 'b's in our thinking and communicating here because we are generalizing about numbers, and this generalizing using letters is a key feature giving mathematical communication a distinctive style – we frequently use few words, and are as concise as possible with symbols.

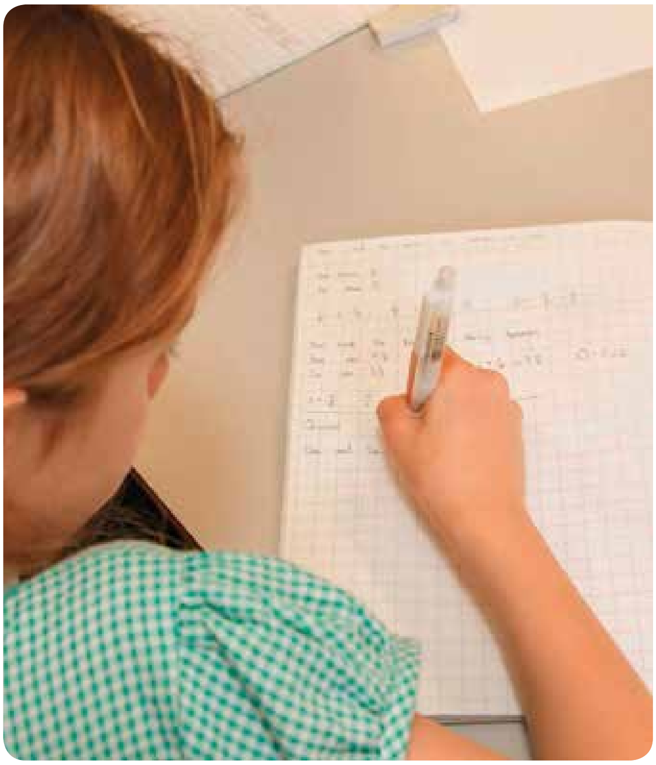
As another example, in the *Number, Pattern and Calculating 5 Teaching Resource Handbook* children were introduced to communicating explicitly and mathematically about 'distributivity', which is a property that explains in general how we can break down a 'long multiplication' into smaller steps. We can communicate this concisely and mathematically however by using the following very short identity in which again a, b, and c stand for *any* numbers:

$$a\,(b + c) = ab + ac$$

For many people, this signals a move into something called 'algebra', but what is actually different is that we have moved up a level of generalization – we are now generalizing about our first generalizations (that is, about our first numbers) – and this requires a new form of communicating: with letters. Notice that if you change the thinking (e.g. in generalizing), you change the communicating (in this case by moving from numerals to letters), and vice versa.

In Number, Pattern and Calculating 6, children are increasingly invited to think about generalizations more

1 This type of equation, which involves values that vary ('a', 'b', and 'c') and yet is always true whatever values the variables take, is called an 'identity'.



explicitly, and to learn to recognize and use different ways in which general statements can be made. Examples might include 'All prime numbers have …', or 'A prime number has …', or 'Any prime number has …'. Children will need to discuss such statements very carefully in order to be clear about what is being claimed, and also to understand the distinctive ways in which we communicate these things mathematically; by looking after children's communicating we look after their thinking.

Importantly, in Number, Pattern and Calculating 6 children are introduced to the use of letters to represent varying known or unknown values; this is usually called 'using letters as *variables*'. This coincides with increasing use of formulae to describe important relationships symbolically, e.g. using $A = l \times b$ to describe how the area of a rectangle relates to the lengths of its sides.

2. **Being systematic**: As we make generalizations in mathematics, we often also make new categories, for example, 'Prime numbers are whole numbers with exactly two different factors'. Then, if we want to find out whether 163 is in that category, for example, we will have to check out all its possible factors before deciding whether it is a prime number or not, and we will have to do this systematically in order to be sure we have looked at *all* the possibilities. When trying to decide whether or not something is possible, or true, in mathematics, we need to make ourselves sure by finding a way of exploring all possibilities *systematically*. (Some useful tests of divisibility that could help you decide whether '163' is prime are also considered in Number, Pattern and Calculating 6.) Making categories is also a vital part of being systematic.

3. **Being logical**: Our generalizations and our categories are crucially involved in our logical reasoning as we do mathematics. A great deal of mathematical reasoning involves following up implications, which are usually expressed as, 'If this is true, then that is also true'. For example, 'If it is impossible to make a rectangular array of more than one row with 11 counters, then 11 is a prime number'. (And we could find this out by being actively systematic – using 11 counters, we can physically try all the possibilities.)

This implication follows from generalizations that 'All composite numbers of counters can be arranged into one or more rectangular arrays involving at least two rows', 'All whole numbers are either composite or prime numbers' and, 'No whole number can be both composite and prime'.

Such reasoning sounds very formal but, as children work on Number, Pattern and Calculating 6 activities, they should be encouraged to follow through 'If … then …' implications in their reasoning wherever possible, and to discuss such implications fully whenever they arise. Illustrations are often particularly helpful in discussions like this, as with all mathematical thinking and communicating.

4. **Contexts**: As children learn to use numbers in an ever wider range of situations, and as they meet new and different kinds of numbers, so they will at times find themselves thinking and communicating mathematically in two quite different kinds of context: in so-called 'real-world' contexts and in mathematical contexts.

In a real-world context, that is, in the kinds of situations we meet in everyday life, the thinking and communicating is about real-world objects and relationships; in a mathematical context the communicating and thinking is about mathematical objects, e.g. numbers. In Number, Pattern and Calculating 6, children will find themselves moving often into mathematical contexts as they meet and discuss new kinds of numbers, and as they begin to use formal algebra.

Children will continue to find connections signalled between the mathematics they are doing at any stage and real world contexts in which that mathematics can be useful. However, they will also spend much more time in Number, Pattern and Calculating 6 calculating with a wider range of numbers and working on how these numbers relate to each other; there are times when they will be working within a purely mathematical context on mathematical objects. Pure numbers, such as whole numbers, fractions and negative numbers, when they are not related to actual amounts and measures, are all simply mathematical objects.

For example, as children learn to treat fractions as objects, that is, learn to talk about '$\frac{3}{5}$' as a thing, and not as '$\frac{3}{5}$ *of*' something, as they learn to convert $\frac{3}{5}$ to 0.6 as a decimal fraction and to 60% as a percentage, as they learn to interpret $\frac{3}{5}$ as 3 ÷ 5, and as they learn to calculate more generally with fractions, they will be thinking and communicating strictly within a mathematical context about how mathematical objects relate to each other. This can be difficult, and many children will often feel such work is too 'abstract'.

It will help during such work, then, to encourage children to move between real-world and mathematical contexts, and to use as many different kinds of illustration as possible. Successful mathematical thinking and communicating at the level of Number, Pattern and Calculating 6 involves being able to move smoothly between real-world contexts and mathematical contexts, to move smoothly from *general* (mathematical) relationships to *particular* (real-world) cases, as well as to move in the opposite direction from seeing a pattern in particular (real-world) cases to making a (mathematical) generalization.

Many children who balk at working out an abstract 7·6 – 2·75 = □, for example, will be able to subtract £2·75 from £7·60 perfectly accurately; these are children who are at the moment more comfortable with real-world objects than with mathematical ones. Exploring a wide range of illustrations will generally help children to connect the two types of objects and the two types of contexts. Number lines are excellent illustrations in a situation like this since physical distances along a line represent abstract numbers; mathematical objects here are directly represented by something physical. Try asking them about where 7·6 and 2·75 appear on a number line and 'seeing' their difference.

A great deal of mathematical thinking and communicating thus involves learning how to move between contexts in two connected but distinct worlds: the world of physical objects and relationships, and the world of mathematical objects and relationships. In Number, Pattern and Calculating 6, children are constantly engaged with this movement between these different worlds as they use many more kinds of mathematical objects; encourage them to illustrate and to talk about such to-ing and fro-ing between 'real' and mathematical objects wherever possible.

Pattern and algebra: essential to mathematics for children of all ages

An essential idea underlying all Numicon activities is that of pattern. Pattern may not sound like a particularly mathematical idea, as we are used to patterns of one sort or another occurring in so many non-mathematical contexts. We could not have learned to speak, for instance, without noticing patterns in the sounds we heard as infants; patterns (rhythms) structure music and dance, and patterns in stories and plays enable us to anticipate the unfolding of a plot (they also, of course, allow our expectations to be manipulated by writers and composers). Much poetry depends upon patterns for its effect, and most scientific research is an attempt to discover or establish patterns in observable phenomena.

Importantly, it is the detection of patterns in our experiences that makes essential aspects of our lives predictable. And since being able to predict successfully is absolutely vital to human survival, seeing patterns is also something humans generally do well. Seeing patterns is what enables us to generalize and then to predict what comes next, thus gaining a degree of control over our environment and our futures.

Patterns are essential in mathematics for a very special reason: they enable us to imagine actions, events and sequences going on 'forever' without us having physically to work out and wait for each and every step. It is patterns that allow us to generalize into the future. Counting is a good example. As we have invented a system for generating number names, we can imagine what it would be to count forever without ever actually having to do it. Most people know they could count to one million, without ever having done it. They know they could because they know the place value system; they know the patterns in number names that would enable them to go on forever. Importantly, once children see the pattern that each next whole number is 'one more' than the previous one, they also know how counting things may go on forever – a vital generalization that allows children to work with collections of any size.

As another example, by noticing the pattern that it doesn't matter which way round you multiply any two numbers (a generalization we call the **commutative property**), children gain the insight that they only have to remember half of their multiplication tables. And because they have generalized in this way, they don't have to keep checking every example.

It is impossible to overestimate the importance of pattern to mathematical thinking. In fact, a very large part of algebra, often thought of as the most powerful branch of mathematics, consists of seeing, manipulating and generalizing from patterns. It is important to remember that

in all the key mathematical ideas discussed here, pattern and generalization are fundamental.

Formal algebra

For many people, doing 'algebra' means 'using letters instead of numbers', and there is some rough sense in this. There are occasions when we want to talk about either a range of numbers (and not one particular number), or about a particular value when we don't know what it is. In both situations, since (for these two quite different reasons) we can't specify *particular* numbers, we use letters to talk about relationships between numbers instead.

Thus sometimes we use letters to stand for *unknown* numbers and sometimes we use letters to stand for a known *range* of numbers. These two very different uses are both technically described as using letters as 'variables' because the value(s) a letter is used to stand for can 'vary'.

Suppose we had 48 square floor tiles to tile along a corridor, that each tile is 40 cm square, and that the width of the corridor is 1 m 20 cm. How far would these tiles stretch along the corridor? We could reason that three tiles will fit nicely across the width, and that 'n' (the number of tiles that would stretch along the corridor) would therefore be related to the other numbers and measurements by the following equation:

$$n = 48 \div 3$$

Then we could say that the actual distance in centimetres ('L') that the tiles would stretch along the corridor would be given by:

$$L = n \times 40\,\text{cm}$$

In this situation we can use the letters 'n' and 'L' to 'hold' our reasoning through the problem; here we use these letters to represent the unknown numbers that we are trying to work out as we set up our plan for working out their values. Encouraging children to express their reasoning using letters to stand for unknown amounts within a problem situation is very important at this stage of their progress; it enables children to express simply and clearly the relationships (and thus the reasoning) they see as crucial to the problem solution. This is using letters as 'unknowns'.

In the Number, Pattern and Calculating 6 Teaching Resource Handbook (Pattern and Algebra 3), children progress from finding unknowns in single 'empty box' number sentences such as □ – 24 = 37, to finding pairs of numbers that satisfy number sentences involving two unknowns. Children will need to think systematically, (see Communicating and thinking mathematically page 42) to explore the possible pairs of numbers that will satisfy number sentences such as △ + □ = 7, or 12 – △ = □. It is important to vary the operations involved as well, and to explore number pairs that satisfy sentences such as □ × 8 = △, or even $a \div b = 12$.

When we need to generalize about a relationship between numbers (or measures) we use letters because we want to stress that we are talking about whole sets of numbers, not just particular numbers. So when we want to express our generalization that 'it doesn't matter which way around you add two numbers, their total will be the same', we can write the following 'identity' (see the section on Generalizing on page 43):

$$a + b = b + a$$

Similarly, when we want to express the **generalization** that relates the area of a rectangle to the lengths of its sides, we write:

$$A = l \times b$$

because we are stating that this relationship works for *any* values of 'A' and 'l' and 'b'. This is using letters as 'variables'; 'A' and 'l' and 'b' each stand for a range of numbers.

We use formulae to calculate unknown amounts; if we know the lengths of the sides of a rectangle and want to work out its area, then in this specific context the letter 'A' represents the unknown amount, and we calculate it by feeding the particular values for 'l' and 'b' into the formula. Thus depending upon any given context, using formulae can invite children to use letters to stand for both 'unknowns' and for generalizing over ranges of values.

It is important to recognize that the formula $A = l \times b$ is also an example of a **function** (see page 47). We could say that 'the area of a rectangle (A) is a function of the lengths of its sides (l and b)', by which we mean that the area of a rectangle *depends upon* the lengths of its sides.

Another kind of situation in which letters used as variables become increasingly important is in describing the rules that lie behind regular sequences of numbers, e.g. the sequence

2, 4, 6, 8, 10 … (see Pattern and Algebra 2). We could describe the rule behind this simple sequence in words by saying something like, 'It starts at 2, and then moves on in steps of 2, for as long as we like.' Or we could say, 'It's the sequence of even numbers, beginning with 2.' These descriptions of the sequence tend to focus on how we move from one term to the next, having started at 2, and are sometimes called 'sequential generalizations' or the 'term-to-term rule'.

But another way of describing this regular sequence involves a different kind of generalization, sometimes called a 'global generalization'. In this approach we describe the sequence by constructing what we call 'the general term' by using letters (usually 'n') to describe *any* term in the sequence in terms of its *position* in the sequence. So in this sequence we could notice that the first term is '2', the second term is '4', and so on, and write this out fully as two parallel, matched sequences:

Position:	1	2	3	4	5	6	7	8 …
Term:	2	4	6	8	10	12	14	16 …

We could then notice that each term in the sequence has a numerical value exactly twice as big as its position number in the sequence, e.g. the 8th term in the sequence has the value 2 × 8, that is, 16. This allows us to generalize and describe 'the rule' for the sequence by saying 'every term is twice the value of its position number'.

Using the letter 'n' (as a variable) to represent any position number, we can now describe the general term for this sequence as '$2 \times n$', or '$2n$'. The rule is: you can work out the value of any term in this sequence by simply doubling its position number.

In many situations, knowing the general term for a sequence is much more helpful than knowing the term-to-term rule. For instance, in this case if we want to know the 213th term in the sequence, we simply double 213 and arrive at '426' as the 213th term. This is a lot more efficient than adding 2, and 2, and 2, and 2 …, from the beginning, until we have added 2, 212 times.

Number, Pattern and Calculating 5 laid some important foundations for children's work on identifying the rules of sequences by creating several series of growing visual patterns with number rods (*Number, Pattern and Calculating 5 Teaching Resource Handbook*, Pattern and Algebra 4). These activities offered children ways of thinking visually about number sequences that will help them in determining both term-to-term rules and general terms, and such illustrating will again be useful in Number, Pattern and Calculating 6, Pattern and Algebra 2 as children learn to translate visual and verbal understanding of number sequences into symbolic expressions.

An increasingly important aspect of work in Number, Pattern and Calculating 6 involves children learning to use letters to express relationships between numbers and unknown quantities of various kinds symbolically. This aspect of children's mathematical communicating – and hence their mathematical thinking – develops significantly at this stage, particularly in relation to the idea of a **function**.

Functions – a special kind of pattern in mathematics

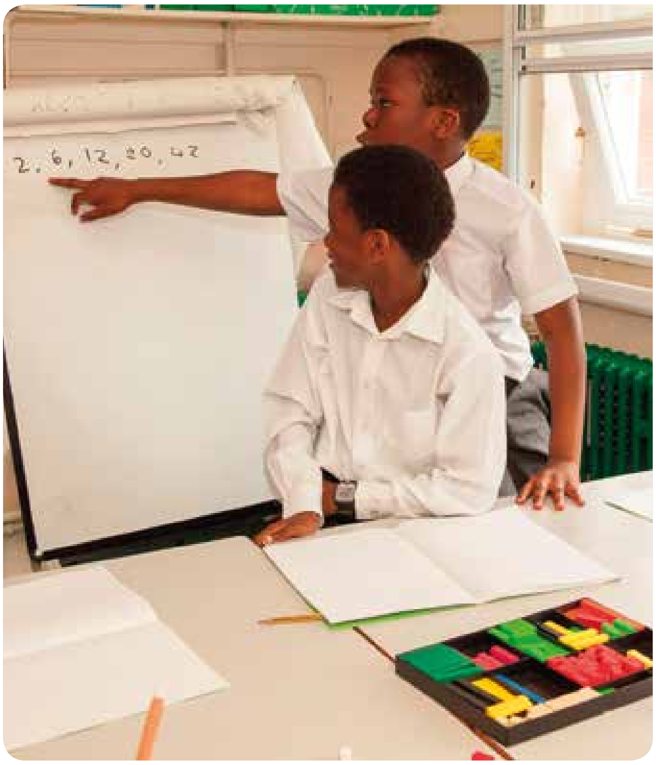
In Number, Pattern and Calculating 6 children continue to work on particular patterns that are called 'functions'. Mathematical functions are used to handle relationships of dependence between changing values, for example, the relationship between time, speed, and distance travelled as we move. The distance travelled at any point depends upon how fast we have been going and how long we have been travelling.

Central to the general idea of a function is that of a 'variable'; this is what mathematicians call whatever it is that is changing. In the travelling example, the variables we speak about are time, speed and distance travelled, all of which change in relation to each other. In Number, Pattern and Calculating 6, children begin to learn to handle functions symbolically with formulae; in the travelling example the formula expressing the function would be $s = \frac{d}{t}$, which could be read as, 'Speed is equivalent to distance travelled divided by time taken.' If you travelled 180 kilometres (d) in 3 hours (t), your average speed (s) was 180 ÷ 3, which equals 60 kilometres per hour. The quantities d, t, and s are said to be 'variables' because they vary, but their interrelationship – the 'function' relationship expressed in the formula, or how they depend on each other – remains the same. Functions are, importantly, another example of generalizations – or seeing the general patterns of relationships.

In our discussion of **formal algebra** (see page 45), we speak about how formulae are used to express relationships of dependence between variables symbolically, both in the example of $A = l \times b$ (in which the area of a rectangle depends upon the lengths of its sides), and in the example of the general term of the number sequence 2, 4, 6, 8, 10, 12 … (The general term '$2n$' defines how the value of a term in the sequence, calculated as $2 \times n$, depends upon its position, n, in the sequence.) We could express this 'general term' rule for the sequence symbolically with the formula $T = 2n$, in which the value of a term, i.e. 'T', depends upon the value of 'n', its position. Alternatively, we could just say in words that 'the value of a term in this sequence is a function of its position in the sequence'.

Because functions are about relationships of dependence, a distinction is sometimes drawn between 'independent' and 'dependent' variables. Sometimes the independent variable is called the 'input' to the function, and the dependent variable is called the 'output'. It is very important to remember that for a relationship to be called a function in mathematics, any given input must determine exactly one unique output (otherwise we wouldn't always know, for example, how 'Area' depends upon 'lengths of sides', or indeed if it always does). If we interpret the formula $A = l \times b$ as showing how the area of a rectangle depends upon the lengths of its sides, then 'l' and 'b' are called the independent variables, and 'A' is called the dependent variable; A depends upon l and b, and for every pair of values of 'l' and 'b' there is just one associated value for 'A'.

In Number, Pattern and Calculating 4 children were invited to generalize in finding a connection between the first few terms of a sequence and the *sum* of those terms, and also to describe the general term of a regular sequence of numbers. At this stage children were encouraged to express their generalizing in words, and use the imagery of number rods to support their describing.



In Number, Pattern and Calculating 5 children began to focus more precisely on what are called 'linear sequences'. Any 'number sequence' in mathematics is simply a collection of numbers set out in an order of some kind; 1, 3, 6, 10… is a number sequence. *Linear* number sequences are so-called because if we plotted a graph with them, that graph would produce a perfectly straight line; this happens only when the numbers involved have a constant difference (or step, up or down) between them.

The sequence discussed previously of 2, 4, 6, 8, 10… is a linear sequence, and if we were to plot the pairs of positions (n) and associated values (T) on a graph we would plot the points (1, 2), (2, 4), (3, 6), (4, 8)…, which all lie along a straight line. Sensibly enough, the **function** relationship that connects values of T and n together ($T = 2n$) is also called a 'linear function'.

In Number, Pattern and Calculating 5 children also considered number sequences with more complicated rules, but at that stage the crucial emphasis was still upon showing these values *visually* (most often with number rods) so that children could literally 'see' how sequences grow, and describe in words how terms grew using visual language. By subsequently assigning number values to the rods involved, the growth of sequences could be described in numerical terms.

With the work on formulae in Number, Pattern and Calculating 6, and on the general terms of sequences, children begin to express generalized relationships of dependence between varying values (**functions**), symbolically. It is always helpful to encourage children to explore relationships visually with illustrations, and to describe relationships in words, before honing their general expression down to concise, symbolic formulae. Functions are a very, very important idea in mathematics, and it is essential that children come to handle these relationships securely.

Patterns in using the four operations: important algebraic relationships

As children begin to calculate in practice, we can help them to notice particular patterns (or connections) in their use of numbers and then to generalize from what they have noticed. Usually we do this without deciding to give these features a formal mathematical name. For instance, we can help children to notice that it doesn't matter which way round we do any multiplying, we will always get the same answer, and then invite them to use that observation to calculate 4 × 7 if they can't remember 7 × 4. We may do that in practice, but we don't often formally call it 'the **commutative property** of multiplying', although we might say 4 × 7 and 7 × 4 are **equivalent**.

However, ideas about how the arithmetic operations work together are important algebraic ideas – they are about number relationships – and consequently within mathematics they do have formal names. Whether or not we think it important that children know and use these formal mathematical names, it is important for children's mastery of calculating that they understand what are called the 'properties' of, and relationships between, the four operations with numbers, especially as they become older and approach formal algebra. In what follows, we use the formal mathematical names for those properties and algebraic relationships that are important at this stage.

Equivalence

Equivalence is one of the most important mathematical relationships of all, and yet it is often the case that not enough attention is paid to it explicitly as we discuss work with children. Children often work with equivalence implicitly from very early on in their thinking, but in doing mathematics at this stage we definitely need to discuss instances of this relationship fully and explicitly, and allow children plenty of time to reflect.

We signal an equivalence relationship in mathematics by using the symbol '=', for example, by writing:

$$\tfrac{4}{5} = 0{\cdot}8 \text{ or } 3 + 4 = 7$$

Equivalence literally means 'equal value'. Quite often the most interesting and important instances of equivalence occur when two or more things are of equal value but look different. In early calculating, there are at least three occasions when children face important instances of equivalence: the introduction of the '=' sign itself (which means 'is equivalent to'), when they encounter quantity value and column value (see **Names for numbers: counting numbers and place value** on page 52), and when they meet fractions, decimals

and percentages ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = 0{\cdot}5 = 50\% = 1 \div 2$ . . .). However, there are numerous other occasions when children need to see equivalence and we are often less explicit.

For their mental calculating strategies to make sense, children have to be able to see what are called the 'decompositions' of any number as equivalent to each other, for example, 9 = 1 + 8 = 3 + 6 = 10 – 1 and so on. Similarly, factor pairs are all equivalent, e.g. 1 × 16 = 2 × 8 = 4 × 4 = 8 × 2 = 16 × 1. In measuring, equivalences between units (100 cm = 1 m, 1″ = 2·54 cm) are at the heart of being able to understand and relate systems of measurement.

When young children seem not to understand something that is clear to us in calculating, there is often an equivalence we see that they do not, or vice versa. We might fully realize that $4 \div 5 = \frac{4}{5}$ for example, before children have come to equate the process of dividing with fractions that are simply 'objects' to them.

There is often also a language problem here. As it may sometimes seem unhelpful to use the formal word 'equivalence' with children, we can often resort to an easier word, 'same', when talking about equivalence; we often say, 'It's the same thing.' Unfortunately, equivalence does not mean quite 'the same thing' – it means **equal value, different appearance**.

In Number, Pattern and Calculating 6, important equivalences between expressions continue to be explored. For example, as children's abilities to calculate with a wider and wider range of numbers develops, success depends upon seeing equivalences between common fractions and decimal fractions, between equivalent proper fractions, between mixed numbers and improper fractions, and between all of these and percentages. It is increasingly important to children's developing understanding and to the fluency of their calculating that they realize '0·5' and '$\frac{1}{2}$' and '50%' and '1 ÷ 2' all look different, and yet they are equivalent ways of referring to the same value.

Equivalence is the key relationship underlying efforts to 'simplify' fractions in Number, Pattern and Calculating 6, for example, being able to simplify $\frac{12}{20}$ to its simplest form of $\frac{3}{5}$ (see also **Fractions** on page 55). Later on in their schooling, children will depend upon this relationship as they are asked to 'simplify' increasingly complex algebraic expressions.

Inverse relationships

Adding and subtracting have what is called an 'inverse relation' to each other. What this means is that each can 'undo' the other. If I add 6 to a number, I can then undo that adding by subtracting 6, and vice versa. This knowledge is important to children for several reasons. Firstly, the more connections children can make between things they learn, the more meaningful their learning is. Secondly, it is important children don't think adding and subtracting are completely unconnected, because if they do they will never understand the 'inverse of adding' structure of subtracting. Finally, children should understand that adding or subtracting can always be checked by doing the inverse calculation: we can always check our adding by subtracting, and vice versa.

Multiplying and dividing also have an inverse relation to each other. Noticing how dividing undoes multiplying (and vice versa) is crucial to connecting these two operations with each other. This will also help children to see that, if multiplying is seen as repeated adding, it makes sense that dividing can be seen as repeated subtracting (quotition). These are important foundations for the extended multiplying and dividing calculations children develop in Number, Pattern and Calculating 6.

As noted (on pages 52–53) in relation to counting and place value, 'partitioning' numbers in various ways while calculating is the inverse action to the 'grouping objects into tens' actions that children have practised regularly in answering 'how many?' questions without counting.

Inverse relationships are also used in relation to the 'empty box' notation that was first introduced in the *Number, Pattern and Calculating Teaching Resource Handbook 1*. Children begin to learn the important algebraic use of symbols to stand for unknown amounts (see above) by using empty boxes. For instance, in asking them to solve 3 + ☐ = 10, we ask them to work out which number should go in the box to make the number sentence true. This type of apparent 'adding problem' requires children either to 'undo' a number fact they can remember, or to subtract 3 from 10; in either case they are using an inverse relationship. Empty box problems also ask of children a clear understanding of **equivalence**.

Although we do not yet make this explicit to children, from the *Number, Pattern and Calculating 4 Teaching Resource Handbook* onwards they have also met inverses of another kind. In negative numbers, children meet what are called the 'additive inverses' of positive numbers. Put simply, $^{-}2$ is the additive inverse of $^{+}2$, $^{-}3$ is the additive inverse of $^{+}3$ and so on. If we add any two 'additive inverses' together, the result will always be 0 (**zero**); in an important sense an additive inverse undoes what its partner can do – and together they are equivalent to 'doing nothing'.

There is an increased use of inverse relationships in Number, Pattern and Calculating 6 as children increasingly depend upon relating factors and multiples to each other, both in 'simplifying' and in developing their calculating with fractions.

In unit fractions, children are also meeting the 'multiplicative inverses' of whole numbers, e.g. $\frac{1}{2}$ is the multiplicative inverse of 2, $\frac{1}{3}$ is the multiplicative inverse of 3 and so on. If we multiply any pair of multiplicative inverses together, the result will always be 1. Any multiplicative inverse undoes what its partner can do: together they are equivalent to 'doing nothing' when multiplying, that is, to multiplying by 1.

We don't expect children working through Number, Pattern and Calculating 6 activities to be discussing multiplicative or additive inverses with you or with each other. What is important is having an awareness of how children are gradually learning more and more about the individual roles of numbers and operations that go together to make up what is called our 'real number system'. It is crucial that children increasingly understand how all the individual pieces – new operations, new kinds of numbers – fit together into this coherent system, and inverse relationships are an important part of that.

Zero and one: examples of 'doing nothing'

Most children notice that there's something funny about zero. Quite rightly, too: there is. Within adding and subtracting, zero is what is called an 'identity element', which means that operating with it leaves everything exactly as it was; adding or taking away zero amounts in effect to 'doing nothing'. Children need plenty of help understanding this because (again, quite rightly) they can't see the point of doing nothing. There is no point, it is simply that zero is a number – it has its own important position on the number line and it can be added and subtracted. It just gives a strange result when added or subtracted: no change at all. When multiplying or dividing, 1 is the identity element; multiplying or dividing by 1 leaves everything just as it was.

Importantly, as multiplying and dividing are introduced the role of zero becomes even more bizarre. In fact, in these operations zero becomes a kind of rogue element, destroying everything it touches. Children will find that multiplying anything by zero always results in zero itself – a very strange result. Even stranger is the fact that dividing by zero is simply not defined in mathematics; it is a calculation with no answer at all. Not many children ask about dividing by zero, although by the time they are working on Number, Pattern and Calculating 6 some may begin to do so; if you are asked, try talking through and exploring with children, 'What would happen if we tried?'

Commutative property

Adding has what is called a 'commutative property'; subtracting does not. It does not matter which way round you do an adding sum; it does matter when you are subtracting. 12 + 6 equals 6 + 12, but 12 – 6 does not equal 6 – 12. Similarly, multiplying (because of its repeated adding structure) is commutative; dividing (because of its repeated subtracting structure) is not.

Associative property

If you have three numbers to add together, it doesn't matter which pair you add first before then adding the third. With 2 + 3 + 5 for example, you can add the 2 and the 3 first, or the 2 and the 5 first, or the 3 and the 5 first. Whatever you do, you always get the same answer: 10. Because of this, adding is said to have an 'associative property'. The same applies to multiplying: try 2 × 3 × 5.

With subtracting this doesn't work. Try it out with 12 – 4 – 1. Is the answer 7 or 9? It could be either; we don't know. Examples like this explain why it is often clearer to use brackets in many numerical and algebraic expressions (see also the **BODMAS** convention on page 57); in this case, using brackets makes the expression clear and unambiguous: (12 – 4) – 1 = 7 and 12 – (4 – 1) = 9.

Similarly, division does not have an associative property. Try 24 ÷ 2 ÷ 3. Is the answer 4 or 36? Quite a difference! Try (24 ÷ 2) ÷ 3 and compare the answer with 24 ÷ (2 ÷ 3).

Once again, following **BODMAS** is necessary to make the original expression clear and unambiguous; if we have several operations of the same kind to work out, we do them in order *from left to right*.

In Number, Pattern and Calculating 4, children were introduced to the use of brackets in order to be clear about expressions such as 6 × 5 – 2 × 5, which we conventionally record as (6 × 5) – (2 × 5) so that children do not work out a value for the whole expression by carrying out the series of operations in the order they read them, from left to right. From Number, Pattern and Calculating 4 onwards, children will have met the first part of the **BODMAS** convention in learning to work out the value of whatever is inside brackets first.

Distributive property

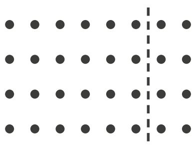
In Number, Pattern and Calculating 6, children continue to make use of a very important relationship between adding and multiplying that helps us to multiply large numbers together quite easily. To give it its full name, 'the distributive property of multiplication over addition' is what allows us to break down a large multiplying calculation into a series of smaller calculations that are easy to do mentally – provided we can recall our tables.

Expressed algebraically, this property could be summed up as:

$$a(b + c) = ab + ac$$

which really means, 'You can do any multiplication a bit at a time, and then add up the individual results to get the final answer.'

The key image that we use to illustrate this property is that of an array. For example:



This array can be used to illustrate that 8 × 4 = 4 × 8 (**commutative property**), and also that

$$4 \times (6 + 2) = (4 \times 6) + (4 \times 2)$$

This is the property that lies behind both traditional written methods of 'long multiplication' and the 'grid method' of multiplication, and of course applies to all numbers, however large. Thus we can break down multiplying 247 by 7 into three more manageable chunks:

$$7 \times (200 + 40 + 7) = (7 \times 200) + (7 \times 40) + (7 \times 7)$$

$$= 1400 + 280 + 49$$

$$= 1729$$

Note that the **distributive property** is also the property that allows us to divide numbers in stages, that is, it is the property that makes possible both short and long division. For example, when dividing 276 by 6, the short division method involves dividing 270 by 6, and then (after exchanging the remainder and adding 6 ones) 36 by 6; the respective results of 40 and 6, when added, give the required result of 46.

Non-computational thinking

All of the patterns and relationships described so far are used in what has in recent times become known as 'non-computational thinking', and this important work continues to be developed in Number, Pattern and Calculating 6. 'Non-computational thinking' is a term that has begun to be used to describe ways of manipulating relationships between numbers *without* actually computing a numerical outcome.

Non-computational thinking is important for at least two reasons: firstly, it helps to lay a foundation for children's formal algebraic thinking; secondly, it is often extremely useful for converting an apparently complex or difficult calculation into an **equivalent** and easier one.

As children move towards formal algebra, they move towards describing relationships between both known and unknown numbers explicitly and symbolically. As an example of something children will need to be able to work with later on, think about the equation $2x^2 - 7x + 3 = 0$, in which x represents an unknown number, or possibly, more than one number.

As children work to solve this equation, that is, to work out what values 'x' might have, they will need to manipulate this combination of known *and unknown* numbers in exactly the same way that we calculate with all known numbers. You may remember that children will need to 'factorize' this equation, that is, to work out a pair of factors that, multiplied together, produce the expression on the left of the equation. Such a pair of factors turns out to be $(2x - 1)$ and $(x - 3)$, so that we can then say:

$$2x^2 - 7x + 3 = (2x - 1)(x - 3) = 0$$

and then deduce that x must either be $\frac{1}{2}$ or 3 because if:

$$(2x - 1)(x - 3) = 0$$

Then *either* $(2x - 1) = 0$ or $(x - 3) = 0$ or both of them do, because the only way in which the product of two numbers can be zero is if either one, or both, of those numbers is itself zero. Hence, if $2x - 1 = 0$, then 'x' must be $\frac{1}{2}$, or if $x - 3 = 0$, then 'x' must be 3.

To do such algebraic thinking, children will need to know how numbers work with each other *in general*. This might be called thinking 'about' number relationships as opposed to thinking 'with' particular numbers (because in this case we don't actually know what all the numbers are). And so a large part of learning to do algebraic manipulation successfully involves coming to know how to manipulate numbers, whatever they are, and in particular when we *don't know* what they are. Non-computational thinking is thus an important introduction to key work on number relationships for children, which lays very important foundations for their later mathematics. The work you do with children in Number, Pattern and Calculating 6 on these aspects of calculating are absolutely crucial for their success in mathematics at secondary school.

Useful work on such non-computational thinking can also be developed as we invite children to think about changing a potentially difficult calculation into an equivalent but easier one, before they try to compute the specific answer. For example, 580 + 260 can be changed into (600 + 260) – 20, which is equivalent and (for many people) much easier to calculate mentally. Children at this stage should always be encouraged to think about any calculation they face *before they calculate*, rather than rushing in to calculate with the particular given numbers straight away. It is always possible that there is an equivalent calculation that would be much easier to carry out.

Tasks specifically requiring non-computational thinking can always be given to children simply to get them thinking in a 'relational' (algebraic) way. For instance, give children a statement like,

$$350 + 280 = 330 + 300$$

and ask them if they can explain why it is true *without* computing either addition calculation. The thinking required is non-computational (we don't want them to do the stated sums) and is also a great deal to do with seeing **equivalence**.

It is important to recognize that successful non-computational thinking underpins the kind of flexibility and fluency in calculating that we want all children to develop as they move into secondary schooling.

Names for numbers: counting numbers and place value

Our civilization has been very clever in devising a system for generating symbolic number names which not only allows us to go on inventing new names for counting numbers 'forever', but which also allows us to tell instantly where in the series of number names any particular name will be found. When we read '273' successfully we know that it is the name of the whole number that comes immediately after 272 and a hundred before 373. This means that we don't have to remember every individual symbolic number name and its place in the order (which would be impossible anyway, since there are an infinite number of them); we just have to master the system that generates the names.

The two essential keys to generating this infinite set of names for counting numbers are that we 'group into tens', and that we use a writing code we call 'place value'.

The first of these keys is **grouping** into tens. The number we call 'ten' (in numerals, '10') is the most important number in our naming system, because, when we are counting collections, as soon as we have ten of something we call them 'one' of something else. So ten 'ones' are called one 'ten', ten 'tens' are called one 'hundred', ten 'hundreds' are called one 'thousand', and so on. In effect, in the language we use, we are always grouping things into tens (and then grouping groups) to call them one of something else.

In children's early experiences of finding how many objects there are in a collection, it was always important to help them group collections physically into tens as they worked to find out how many things they had before them. Finding 'how many?' by grouping in tens (and then tens of tens) reminded children that our way of naming numbers uses a ten-based system, and this idea remains crucial to their understanding of the calculating techniques developed in Number, Pattern and Calculating 6, including long multiplication and division.

The second key to our symbolic numeration system is **place value**, which is a kind of shorthand describing how the place of each digit within a string of digits signifies an important value. So it is the place of '2' in the string '427' that tells us it has a value of 2 tens, or 20. It is important to realize, then, that the term 'place value' actually refers to a symbolic code for naming and reading number names, and that children have to learn either to crack the code or to reinvent it for themselves (depending on how they are taught).

Some people usefully distinguish between what is called the 'column value' and the 'quantity value' of a digit. For example, the column value of '2' in '427' is '2 tens', because it is in the 'tens' column, while its quantity value is '20', because that is its value as a quantity. In Numicon, we feel the important thing is that children understand that column value and quantity value are **equivalent**, that is, that the '2' in '427' means both '2 tens' and '20'; the two values are interchangeable. Children learn this equivalence through joining in our conversations around place value – this is another instance of the vital importance of our conversations with children when teaching mathematics.

The fact that children have spent time grouping objects in tens to 'find how many' in their early stages helps children subsequently to **partition** numbers as part of many calculating techniques. For example, seeing 236 as '200 and 30 and 6', focusing upon quantity value, can sometimes (though not always) be the most helpful way of seeing the number. In essence this partitioning is 'undoing' the grouping that they managed earlier.

Place value in Number, Pattern and Calculating 6

Children are asked to extend their understanding of place value in Number, Pattern and Calculating 6 in two main ways: by increasing use of formal written column methods of calculating, and by extending their calculating to working with both larger and with longer numbers (that is, larger values and values expressed up to three decimal places). Both of these aspects of number work depend crucially upon generalizing from a fundamental understanding of early 'grouping in ten' activities, and of our 'place value' code for number notation.

As in Number, Pattern and Calculating 4 and Number, Pattern and Calculating 5, process terms for grouping and re-grouping in tens, hundreds and so on, such as 'carrying', 'exchanging', 're-distributing' and 're-grouping' are used explicitly. Partitioning and recombining numbers in these ways as children calculate are another instance of 'doing and undoing' actions – a key element of mathematical thinking

for children to develop in many aspects of calculating (discussed further in the section on **inverse relationships** on page 49). Children's use of numbers up to 10 000 000, and increasing emphasis upon decimal fractions in Number, Pattern and Calculating 6 also places increasing demands upon their ability to read, order and position numbers between other numbers of any size.

As work with decimal fractions extends now to multiplying and dividing numbers with decimals, including multiplying and dividing by multiples of 10, 100 and 1000, children come increasingly to recognize and use the constant **ratio** relationships between 'places' (or columns) in our system of number notation, however large or small the numbers. That is, children increasingly recognize and use the regular '× 10' and '÷ 10' relationships between column values as we move to the left and to the right respectively in reading, writing and calculating with multi-digit numbers. The significance of the decimal point marking out 'fractions' to the right of it continues to be reinforced as children also practise rounding, and estimating the answers to calculations before calculating.

Negative numbers

Negative numbers are a very old idea, explicitly discussed and used in Hindu writings of the seventh century CE and in much earlier Chinese calculating, where the colours of 'coloured rod' images denoted either a positive or negative aspect. In both cases the context was accountancy, with negative numbers signifying an amount of debt. Calculating with these numbers was always dependent on the agreed sense they made within the practical contexts of money, loans, assets and debts.

Interestingly, for centuries many European mathematicians resisted the idea that negative numbers are as valid mathematically as 'natural' (counting) numbers, refusing to allow that negative solutions to equations could be meaningful. Children having difficulties with negative numbers today have history on their side.

In Number, Pattern and Calculating 4, following the early Chinese and Hindu examples, children were introduced to negative numbers in practical contexts in which they make sense today – in our case, temperature and underground car-parking levels. The important thing, in both contexts, is that in a real way amounts 'below zero' make intuitive and practical sense. More generally now, the idea of negative versus positive amounts often works in situations where there is some key point of reference that has meaningful amounts on either side of it, for example, years before or after a key event on a timeline, or travelling towards or away from a geographical 'zero' position on a physical line.

The essential thing for children to understand as they meet negative numbers is that from now on numbers may be considered to have not just a 'size' but also a 'direction'; this is why integers (positive and negative whole numbers including **zero**) are sometimes together called 'directed numbers'.

Importantly, using a '**zero**' point on a physical line is what prepares children for the imagery of negative numbers presented as distances to the left of zero on a conventional number line. In Number, Pattern and Calculating 5, children continued to meet negative numbers in context, count forwards and backwards along a number line using both positive and negative numbers, and began to calculate differences between two directed numbers.

In Number, Pattern and Calculating 5, children were also introduced to the important idea of *ordering* directed numbers, and this raised the intriguing question of whether $^{-}12$ is 'bigger' or 'smaller' than $^{-}7$? In Number, Pattern and Calculating 6 we continue to suggest that it is always best to answer such questions in context, so that we could say $^{-}7$ °C is 'warmer' than $^{-}12$ °C, and that someone who owes £7 is 'better off' than someone who owes £12. Otherwise it is still probably best to focus simply on direction when ordering directed numbers and to ask, e.g. 'Which is *to the right of* the other one on a number line?' In Number, Pattern and Calculating 6 children continue, in context, to calculate intervals that cross **zero** on a number line.

Factors and multiples, prime and composite numbers, squares and cubes

As children learned their multiplication tables earlier and thereby gradually became more familiar with multiplicative relationships between numbers, they were encouraged to make – and to explain – observations such as '36 seems to crop up in lots of places in the tables'. A typical explanation

would be because 36 is the product of 3 × 12, 12 × 3, 4 × 9, 9 × 4, and of 6 × 6. We were able to use observations like this to introduce children to the term **factor**, meaning a number that divides into another number exactly, without leaving a remainder. So 3, 4, 6, 9, and 12 are all called factors of 36 because they divide exactly into 36 without leaving a remainder; 2 and 18 are also factors of 36, but this is not obvious from multiplication tables up to 12 × 12. By introducing the term 'factor' we could encourage children to explain their observations with, for example, '36 seems to crop up so many times in the multiplication tables because it has a lot of factors'.

Afterwards we were also able to put the relationship the other way around and say that since 2, 3, 4, 6, 9, 12, and 18 are all factors of 36, this means conversely that 36 is a **multiple** of 2, 3, 4, 6, 9, 12, and of 18.

Not all numbers have as many different factors as 36 does; some numbers have only two different factors, and such numbers were introduced as **prime numbers**. 3, 5, 7, 11, 13, and 17 are all prime numbers; each of these is divisible only by 1 and by itself, e.g. 3 is divisible only by 1 and 3. '1' is not normally called a prime number because it only has one factor, that is, 1 itself. 2 is the only even prime number.

Any positive whole number that is not a prime number is said to be a **composite number**. 36 is a composite number because it is a positive whole number, and it has many more than two different factors.

Any composite number can also be broken down into its **prime factors**. Working out the prime factors of any number, however, involves more than just listing those factors of a number that are themselves prime numbers, e.g. establishing that the only factors of 12 that are prime numbers are 2 and 3. **Prime factorization** means working out how to express any composite number as the *product* of prime factors, so for example, the prime factors of 12 = 2 × 2 × 3 or $2^2 \times 3$.

By presenting composite numbers as products of their prime factors, we can make some calculations easier, or more systematic. In particular, 'products of prime factors' make finding the Highest Common Factor (HCF) of two numbers, or their Lowest Common Multiple (LCM) easier, and also the reducing of common fractions to their lowest terms.

Factors and multiples, prime and composite numbers, squares and cubes in Number, Pattern and Calculating 6

In Number, Pattern and Calculating 6, children are invited to make use of their increasing familiarity with tables facts (factors and multiples), as they learn to 'simplify' fractions, to convert improper fractions to mixed numbers (and vice versa), and to calculate equivalent fractions in order to compare them, order them, and to add and subtract them from each other.

Children continue to recognize that when a number is multiplied by itself, the product is said to be a **square number** (probably because the area of a square is calculated by multiplying the length of its side by itself). As an example, 36 is called a square number since it is the product of 6 × 6.

Children also continue to recognize that when a number is multiplied by itself twice, the product is said to be a **cubic number** (or a '**cube**'), probably because the volume of a cube is calculated by multiplying the length of its side by itself, twice. For example, 8 is a cubic number since it is the product of 2 × 2 × 2.

In Number, Pattern and Calculating 5 children also were also introduced to tests of what is called 'divisibility'; these are ways of testing whether a number has a particular factor, or not. For example, a number has a factor of 4 if the last two digits are also divisible by 4. Children continue to explore such patterns in Number, Pattern and Calculating 6 as part of their continuing work on recognizing factors, multiples, and prime numbers more fluently.

Fractions, decimals, ratios, proportion and percentages

Fractions occur as part of a complex set of relationships, and confusingly for many children there are also several different symbolic ways of representing what are essentially the same numbers, e.g. $\frac{3}{5} = \frac{9}{15} = 0{\cdot}6 = 60\% = 3:5 = 3 \div 5$. One of the key challenges for teachers at this stage is to guide children to understanding that common fractions, decimal fractions, percentages, ratios, proportions and dividing calculations are essentially different forms of notation for expressing the

same 'rational' numbers, and that **ratio** is at the heart of what is called overall, **multiplicative thinking (**see page 64**)**.

Typically for children, fractions *of* things arise in measuring situations, which importantly include 'sharing'. The measuring of continuous quantities, such as time, length, or chocolate, is always approximate and for this reason we commonly find ourselves needing parts of whole units to describe amounts accurately. The moral imperative for fair shares usually draws children easily to the view that fractions are, and indeed should be, about *equal* parts (or **proportions**) of a whole (although in Number, Pattern and Calculating 6, as children study proportion, they will also meet instances of unequal sharing).

The two main ways in which children communicated about fractions in early Numicon activities were as 'operators' and as 'descriptors' – using fraction words as verbs and as adjectives. An initial invitation to '*halve* twenty-six' would be an invitation actively to find 'half' of 26 – the fraction word is used initially as part of an instruction to *do* something. Then, to describe the outcome of some measuring tasks, or of some dividing calculations, children would use fraction words as adjectives, for example, in the description 'twenty-six-and-a-half *somethings*', or as the description of a relative distance, for example, as 'halfway' between 26 and 27 on a measuring scale.

It is important to note too that in work from the N*umber, Pattern and Calculating 2 Teaching Resource Handbook* onwards, children were also meeting fractions as *objects* (that is, as 'numbers' in themselves), signalled by the use of fraction words as *nouns* and by their representation along a pure number line. From introducing halving situations in which there was an action of halving (say) a pizza, and in which children used the verb, 'to halve', we moved to describing actual amounts of things using fraction words as adjectives; then we subsequently asked them to do a very strange thing, which was to start talking about 'a half' as a rather isolated abstract mathematical object, using the word 'half' on its own, as a noun. Note that in a mathematical context, in which fraction words are used as nouns, any trace of pizza (or of anything else from a material world) has disappeared altogether; the key illustration that supports children thinking about '$\frac{1}{2}$' as an abstract mathematical object is usually the distance along a number line between 0 and the 'half-way' point between 0 and 1.

It cannot be emphasized too strongly that talk of actions (using verbs), turning to talk of 'fractions *of* something' (using adjectives) turning to talk of just isolated 'fractions' as mathematical objects (using nouns) involves very significant changes in our communicating for children to join in with. The models and imagery we offer children to help them get used to these strange new aspects of our communicating are crucial, and once again children need plenty of time, opportunity and imagery to get fully used to our new ways of talking and communicating about them.

When using fraction words as nouns and beginning to ask children questions such as '$\frac{2}{5} + \frac{4}{5} = ?$', it can be helpful to suggest that children read the number sentence as, '$\frac{2}{5}$ *of anything* + $\frac{4}{5}$ *of anything* = what?' This may help children to understand that fraction words and symbols used on their own, as nouns, are **generalizations**, and that '$\frac{2}{5}$' used on its own means '$\frac{2}{5}$ *of anything*'.

In Number, Pattern and Calculating 1 and 2, common (or 'vulgar') fractions were introduced with their conventional notation (e.g. '$\frac{2}{3}$'), and as new number objects they began to be related to existing whole numbers through also representing them as distances along a number line. In Number, Pattern and Calculating 3, the terms 'numerator' and 'denominator' were formally introduced, counting on and back in fractions along a number line was further developed, and some fractions (< 1) with the same denominator were **added** and **subtracted**.

In Number, Pattern and Calculating 4, key developments involved the introduction of decimal fractions, mixed numbers and improper fractions and, importantly, recognizing the **equivalence** of a range of common fractions (< 1) e.g. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} \ldots$, which is fundamental for later methods of adding and subtracting fractions with different denominators.

Importantly, strong explicit connections were made in Number, Pattern and Calculating 5 between common fraction notation and the process of dividing, that is, recognising that $\frac{5}{8}$ is **equivalent** to $5 \div 8$. It can, however, take some time for children to accept an equivalence between an action (dividing) and an object (a fraction), and it will be useful still to allow plenty of discussion about this in Number, Pattern and Calculating 6 as well.

We discuss the important relations between the **fractions, ratios**, and **proportions** considered in Number, Pattern and Calculating 6 in the section on **multiplicative thinking**, page 64.

Fractions, decimals, ratios, proportion and percentages in Number, Pattern and Calculating 6

In Number, Pattern and Calculating 6, children continue to develop their abilities to translate fluently between the various forms of fraction notation: between equivalent common proper fractions, between mixed numbers and improper fractions, and between common fractions, decimals, percentages, and division calculations. This developing flexibility of expression allows children now to add and subtract fractions with different denominators.

Significant further developments in Number, Pattern and Calculating 6 are the introduction to multiplying proper fractions, and to dividing proper fractions by a whole number. The illustrative use of arrays and rectangles is crucial to children's understanding of how to multiply proper fractions together, and the dividing of proper fractions by whole numbers can be well supported by connections with the dividing of decimal fractions by whole numbers.

The use of Numicon Shapes, number rods and objects arranged in arrays, and visual imagery (such as diagrams and number lines) continues to be essential to communicating about fractions, as is the use of everyday and realistic contexts that children can relate to. Measuring scales are particularly useful. When using the terms 'numerator' and 'denominator' with older children it can also be helpful to explain their sense. A de**nom**inator gives a common fraction its name – it tells you what kind of a fraction it is. A **num**erator tells you how many of this kind of fraction you have. There is always a history to how we do and say things in mathematics.

Complex expressions involving arithmetic operations – BODMAS

As the mathematical problems children face become more complex, so calculating becomes more complex in that several arithmetic operations can be combined together in one sequence. For example,

$$T = 5\,(3^2 + 8 \times 7) \div 6\,(4 + \sqrt{9} \times 8)$$

Since the above expression for 'T' could be simplified, or T's value could be calculated, in a number of different ways that would all give different answers, there is an agreed international convention that determines the order in which the various operations involved are to be tackled, so as to avoid ambiguity. In the UK, this convention is usually remembered with the help of the acronym BODMAS, which identifies the sequence of operations as:

Brackets (do everything inside brackets first)

Order (that is, powers and roots)

Division and **M**ultiplication

Addition and **S**ubtraction

Also, when two or more operations of the same order appear one after another, the operations should be carried out from left to right. So,

$$24 \div 3 \times 4 = 32 \text{ (not 2).}$$

So, we would begin to deal with the expression for T by looking inside the brackets first (**B**), and within the brackets dealing first with powers and roots (**O**). Purely for the purposes of this example, if we take $\sqrt{9}$ as $^{+}3$ (and ignore the possibility of $^{-}3$ as another square root), this gives us:

$$T = 5\,(9 + 8 \times 7) \div 6\,(4 + 3 \times 8)$$

We can then finish off the work inside the brackets by doing the multiplying (**M**) before we do the adding (**A**):

$$T = 5\,(65) \div 6\,(28)$$

Which would then become:

$$T = 325 \div 168$$

As a division, this can then be resolved into whatever form suits the context of the original problem – either as an improper fraction, a mixed number, a decimal fraction, or leaving a remainder.

BODMAS priorities also resolve some quite subtle associated ambiguities that become more important as children progress, for example: -5^2 is conventionally interpreted to mean $^{-}(5^2)$, meaning 'do the power first', giving a value of $^{-}25$,

rather than $(^{-}5)^2$ which would be 'do the negative first', giving an answer of $^{+}25$.

Two further aspects of the BODMAS convention are important for children's future progress: not all electronic calculators (or indeed programming languages) follow this convention (some calculators just do the operations in the order you key them in), and this convention underlies the interpretation of algebraic expressions and formulae that children will meet later in secondary school, such as the formula for solving quadratic equations:

$$x = \frac{^{-}b \pm \sqrt{b^2 - 4ac}}{2a}$$

The order in which these calculations are done will usually make a big difference to the result.

Arithmetic operations, or 'the four rules': adding and subtracting

In Number, Pattern and Calculating 6, children continue to develop their adding and subtracting of whole numbers, with an emphasis upon written methods necessary for calculations too difficult to be accomplished purely mentally. Note that all written calculating involves an element of working mentally (remembering number facts and so on), and children should therefore also practise mental arithmetic continually.

Even though much focus is upon developing written column methods of calculating in Number, Pattern and Calculating 6, children should also always be encouraged to think about any calculation first before simply diving in thoughtlessly with

the first method that occurs to them. **Non-computational thinking** may often reduce an apparently difficult calculation to a much easier one that can be carried out purely mentally. Much practice at consciously thinking about how to approach any particular calculation before diving in is essential at this stage, as children develop the flexibility in their calculating that is necessary for fluency.

This is an important point and worth emphasizing: the use of base-ten materials is purely to *illustrate* number **equivalence** relationships in ways that reflect our **place value** system of naming numbers. Actions with the materials are used therefore simply to help children think about how the 'grouping' or 'exchanging' actions involved in most written methods *make sense*, not as a method of 'calculating with blocks' in itself. Actions with materials are not carried out to *produce* an answer, but to *explain* the number actions involved in calculating.

In Number, Pattern and Calculating 6, children build on their knowledge of adding and subtracting both common fractions and decimal fractions, and these operations continue to rely upon children's understanding of both **equivalence** and of **place**, as they now work on adding and subtracting common fractions with *any* denominators.

With all kinds of numbers, with positive and negative whole numbers, and with common and decimal fractions, children continue to work on the following aspects of adding and subtracting in the Number, Pattern and Calculating 6 activities:

- **structures** – the different kinds of situations in which adding and subtracting occur; and
- **methods** – how to calculate.

Structures for adding and subtracting

Within Numicon, we address two adding structures – **aggregation** and **augmentation** – and children should be given regular experiences with both forms in a variety of contexts.

Aggregation is putting together. Two or more amounts or numbers are put together to make a 'total' or 'sum'. For example, 'I had £20. John gave me £10 and Nana gave me another £35'. How much did I have in total?'

Augmentation is about increase. One amount is increased or made bigger. For example: 'Special offer! One third extra free!'

We expect children to recognize four subtracting structures: **take away**, **decrease**, **comparison** and **inverse of adding**. Subtracting is more complex than adding as it is more varied in the different kinds of situations in which it occurs. Again, children should be given experience with all four structures regularly.

Take away refers to those situations where something is lost, or one thing is taken away from another. For example, 'Gemma had £19 at the beginning of the day. She spent £6·47. How much does she have now?'

Decrease is about reduction. For example, 'Special offer! 25% off!'

Comparison occurs when two amounts are being compared and we want to find the additive difference. For example, 'Samir has saved £34·40 and Nihal has £42·65. What is the difference between the amounts of money that Samir and Nihal have?'

As comparisons involving negative numbers are addressed in Number, Pattern and Calculating 6, comparisons and differences between ranges of positive and negative numbers are most effectively illustrated using a continuous number line.

The **inverse of adding** structure is about wanting to know how much more of something we want or need in order to reach a particular target. For example, 'The blue trainers cost £59·50. I have £38·25. How much more do I need to buy the shoes?' Children can often feel very confused about adding on in order to accomplish subtracting, and it is important for teachers to be clear about what is going on here. The reason this adding manoeuvre is included as a subtracting structure is because the adding on in these cases is done in order to find out a *difference*; in most adding we know how much to add, and we do it.

Methods for adding and subtracting in Number, Pattern and Calculating 6

There is continuing emphasis in Number, Pattern and Calculating 6 on developing written methods of adding and subtracting with larger numbers, but children should always be encouraged to think before they act. In particular, children should always be encouraged to think first about whether any given calculation could be transformed into an easier, equivalent calculation, and secondly to estimate what the answer is likely to be – approximately – before calculating.

Non-computational thinking should thus become a habit as children are increasingly asked to think about their calculating, rather than just responding mechanically to an addition or subtraction sign. Is there the possibility of altering a calculation to an equivalent 'easier' one, for instance altering 840 – 380 to (840 – 400) + 20? (Adjustments of this particular kind are sometimes called 'rounding and compensating'.) Children should regularly be encouraged to notice how useful the basic number facts to ten are in calculating with larger numbers, for example in being able to generalize from 6 + 4 = 10 to 60 + 40 = 100, and to 600 + 400 = 1 000.

Non-computational thinking and estimating answers to calculations in advance should of course become a habit for children when using *any* kind of numbers, but especially when adding and subtracting fractions and decimals. This will help to develop their understanding of these numbers enormously.

When adding and subtracting fractions children will find that, just as with currencies of different denominations, it is not possible to add (or subtract) fractions of different denominations. In the same way that we cannot add or subtract \$6 and £3 together directly, so we cannot add or subtract $\frac{4}{5}$ and $\frac{3}{8}$ together directly; in both cases we have to transform the amounts into *equivalent* amounts in a *common* denomination. We either have to convert \$6 to GBP (or £3 to \$), or convert both to a common third currency (e.g. €) before we can add or subtract them; with fractions we would normally convert each of the two above to equivalent numbers of fortieths (their lowest common denominator).

When adding and subtracting numbers involving decimal fractions, children's existing understanding of the place value notation they use for naming whole numbers will be the basis from which they *generalize* to bring meaning to the columns to the right of the decimal point. In particular, children will need to be clear that the 'value' of a column 'place' divides by ten each time we move one 'place' to the right. Thus the columns to the right of the decimal point have values of one tenth, one hundredth, one thousandth, and so on as we move to the right.

In Number, Pattern and Calculating 5, children began usefully to generalize the technique of '**bridging**' through multiples of ten to bridging through larger multiples (e.g. 100), and also to bridging through different kinds of convenient 'whole' points in different contexts. For example, when adding $\frac{4}{5}$ and $\frac{3}{5}$ together we can mentally partition the $\frac{3}{5}$ into $\frac{1}{5}$ and $\frac{2}{5}$, add the $\frac{1}{5}$ on to the $\frac{4}{5}$ first, and thus 'bridge' through 1 to obtain the answer '$1\frac{2}{5}$'. Note that we could also 'bridge through 1' if we were doing the same calculation with decimal fractions: '0·8 + 0·6' can

be managed as 0·8 + (0·2 + 0·4), which is the same as (0·8 + 0·2) + 0·4, which then gives the total '1·4'. Bridging was thus introduced as an invaluable mental technique for adding and subtracting in a wide range of contexts, and particularly when using the units of various measures as bridging points; when adding 350 m to 1 km 800 m, for example, it is helpful to 'bridge' through 2 km. This technique should be regularly encouraged in Number, Pattern and Calculating 6 as it can add significantly to children's calculating fluency, particularly when problem solving.

Arithmetic operations, or 'the four rules': multiplying and dividing

In Number, Pattern and Calculating 6, work also continues on straightforward multiplying and dividing situations and calculations with larger whole numbers, using both purely mental and written methods. As with adding and subtracting, children should be encouraged always to think before they act; **non-computational thinking** will often reveal ways of making an apparently difficult multiplying or dividing calculation much simpler, and children should also get into the habit of always estimating what an answer is likely to be, approximately.

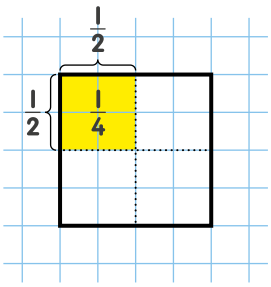
The calculation 36 × 25 = □, for example, can be re-interpreted as (9 × 4) × 25 = □ using a particular pair of **factors** of 36, and then as 9 × (4 × 25) = □ (using the **associative property**), which is then easy to calculate purely mentally. In this case the non-computational recognition and use of factors removes the need for any laborious form of 'long' written multiplication.

Also, as with children's work on adding and subtracting whole numbers, base-ten apparatus is often used in Number, Pattern and Calculating 6 to illustrate the number relationships involved in 'partitioning', 'exchanging', 'grouping' and so on that feature within some written methods of multiplying and dividing with larger numbers. It is again important to establish that base-ten materials – as with all other materials and imagery used in our approach – are used to *illustrate relationships*. Actions with physical materials are not used as methods of producing answers, but to *explain* the sense of number actions that are used in calculating with figures on a page.

By the end of Number, Pattern and Calculating 6, children should be accomplished in using both long multiplication and long division column methods effectively; when either the numbers involved are too awkward, or when non-computational thinking cannot translate a multiplying or dividing calculation into an easier form, children should be able to employ both long multiplication and long division methods smoothly. This takes both understanding and practice.

In Number, Pattern and Calculating 5, children also began to multiply whole numbers with both common fractions and with decimal fractions, and this placed considerably more demand upon their **multiplicative thinking**. 'Multiplicative thinking' is a term increasingly used now to refer to a whole set of different ways of thinking about *comparison relationships*, and is usually contrasted to 'additive thinking' (see **multiplicative thinking**, page 64).

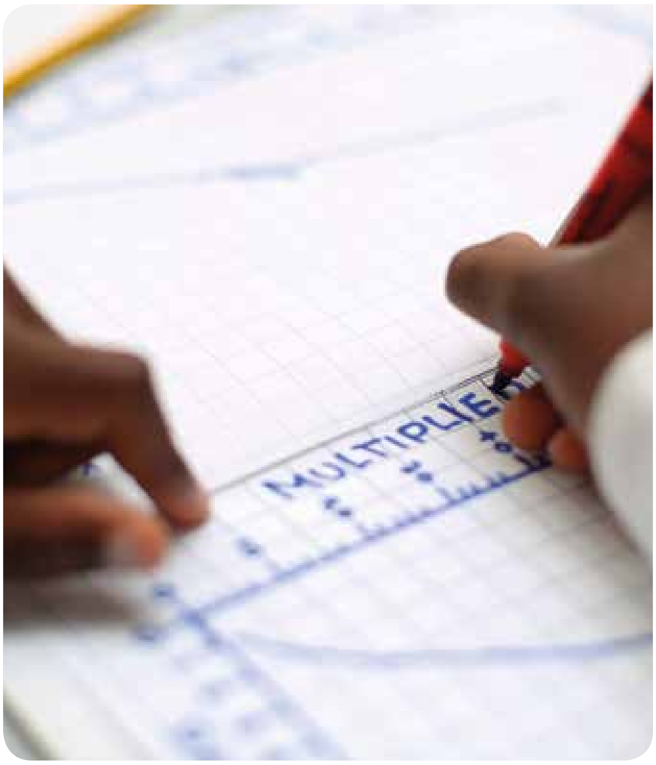
In Number, Pattern and Calculating 6, children are asked to multiply proper fractions together, and also to divide proper fractions by whole numbers. Multiplying proper fractions together often produces surprises for children, who are used to multiplying whole numbers together and obtaining larger products. Many children cannot understand how, for example, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, in which the product is smaller than either of the two numbers being multiplied. The key illustration supporting this work is that of a rectangular array, most helpfully drawn on squared paper. This illustration also shows nicely how children can interpret $\frac{1}{2} \times \frac{1}{2}$ as $\frac{1}{2}$ of $\frac{1}{2}$.



Dividing proper fractions by whole numbers produces smaller outcomes, and this seems generally more acceptable intuitively.

Multiplying and dividing

The operations of multiplying and dividing have several aspects, not all of which make coherent sense to children for



a considerable time. In Number, Pattern and Calculating 6, we continue to distinguish between the **repeated adding**, **ratio** (or **scaling**) and **array** structures of multiplying. Repeated adding and scaling up are often fairly intuitively understandable and build on children's earlier experiences of counting on in 2s, 5s and 10s, and doubling in Number, Pattern and Calculating 2 and 3. The array structure becomes increasingly important in Number, Pattern and Calculating 6 as children learn to generalize their multiplying to a wider range of numbers, particularly to fractions.

As with our discussion of adding and subtracting, in relation to multiplying and dividing we address:

- **structures** – the different kinds of situations in which multiplying and dividing occur; and
- **methods** – how to calculate.

Structures for multiplying

Repeated adding is the familiar 'so many lots of something' idea, in which repeated equal amounts are added. For example, '5 tables each need 6 place settings. How many place settings are needed altogether?'

Ratio is the 'multiplying up' idea we use when we want to scale something up, for example, making a recipe for 6 people instead of 2.

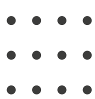
Both of these structures have very important and strong **inverse** connections with **dividing**. Scaling up, for instance, is associated with its inverse in dividing, of scaling down – for example, halving.

Importantly, children should be encouraged to notice that when two numbers are being multiplied together to give a product, in many situations each number plays a different role – one number refers to an amount being multiplied (technically, this is the ‘multiplicand’) and the other determines how many times that number is to be multiplied (the ‘multiplier’). One number is being multiplied; the other does the multiplying. In practice, we teach children pretty quickly that multiplying has a **commutative property** so, for example, 4 × 6 = 6 × 4 and that in a sense it doesn’t matter which number is doing the multiplying, as the product will be the same.

However, this practice of saying ‘it doesn’t matter which one is the multiplier’ can turn out to be unhelpful to children when they later try to make sense of the **inverse** connections between multiplying and dividing. Seeing dividing as the inverse of multiplying (and using our multiplication tables to solve dividing problems) is a bit like turning a dividing problem around and saying, ‘We already know the product of two numbers, but we only know one of the numbers that were multiplied.’

And in practical situations, it can make a difference whether the number we know is the multiplicand or the multiplier. If we know the multiplicand (the size of the groups), we then want to know how many times that number goes into the product; if we know the multiplier, or how many times something goes into the product, we want to know how big that ‘something’ (the multiplicand, or size of the group) was. The first case applies to dividing situations like working out how many 15-seater minibuses we need to ferry 60 children around (this is called ‘quotition’). The second applies to sharing situations such as ‘How much will we each get of that cake?’ (this is called ‘partition’).

The third multiplying structure, that of an **array**, will help children to see the **commutative** property of multiplying. In Number, Pattern and Calculating 6 it also helps them to connect multiplying with the measurement of area, to understand how multiplying by fractions makes answers smaller, and to understand how (in long multiplication) the multiplication of large numbers can be broken down into smaller calculations (the **distributive property**). All of this involves interpreting multiplying as an ‘array’, for example illustrating 3 × 4 as:



In Number, Pattern and Calculating 4, we placed increased emphasis on arrays as children worked to generalize the distributive property – that is, that multiplication is ‘distributive over addition’. It is this property that underlies the ‘grid method’ of multiplication that was also introduced at that stage (see the section on the **distributive property**). Arrays are also helpful to children when facing **correspondence problems** such as, ‘I have 5 T-shirts and 3 hats; how many different outfits can I put together?’ (see **multiplicative thinking**, page 64).

The **ratio** structure of multiplying was developed in the *Number, Pattern and Calculating 3* and *4 Teaching Resource Handbooks* in terms of everyday situations that required some scaling up, for example, of recipes. In order to establish essential links with dividing from the beginning, the images and patterns developed in Numicon activities for multiplying are usually quickly exploited to illustrate dividing; recipes are also scaled down to offer a context for the **ratio** structure of dividing.

All the multiplying structures continue to be relevant in Number, Pattern and Calculating 6, especially as children are increasingly asked to solve problems in a variety of contexts, but (as noted above) the array structure is probably the illustration that best supports the key developments in Number, Pattern and Calculating 6.

Note on a reading convention: When reading and recording multiplying sentences such as ‘4 × 7 = 28’, there are many choices of interpretation and often a surprising amount of controversy about whether ‘4 × 7’ *really* means ‘four 7s’ or ‘seven 4s’. Of course, the array structure quickly demonstrates that their product is the same, but some teachers feel that only one reading of the sentence can be ‘mathematically correct’.

The truth is that we do have choices, and that there are equally good reasons for choosing either way. In Numicon activities, we have chosen to introduce reading ‘4 × 7’ as ‘four times seven’, meaning four lots of seven, for a number of reasons: in order to exploit the everyday use of the word ‘times’ (signalling repeated actions); to tie in with the traditional way of reading and saying multiplication tables in the UK; to be consistent with conventions for units of measure, for example with the meaning of 3 kg as three 'lots of' a kilogram; and to be consistent with algebraic expressions such as $4x + 3y$ (commonly interpreted as '4 lots of x' and '3 lots of y').

Structures for dividing

There are three essential structures of dividing: the **grouping**, **sharing**, and **ratio** (or **scaling down**) structures.

The **grouping** structure – technically called quotition – occurs in situations where we know an amount, the dividend, and we want to know how many times a different amount, the divisor, will go into it. This type of situation will lead to remainders when the divisor is not a **factor** of the dividend. For example, 8 goes into 43 five times, leaving a remainder of 3. We often call this '8 divided into 43' (or '8s into 43'). It is the grouping structure that underlies the form of long division described as 'chunking'.

The **sharing** structure – technically called partition – occurs in situations where, again, we know the amount to be shared, the dividend, but this time we know how many equal parts the dividend is to be shared into but we don't know how big each share will be. This type of situation will lead to **fractions** when the number of shares, the divisor, is not a **factor** of the dividend and the object(s) being shared can be broken into parts. For example, 3 chocolate bars shared between 2 people will give each person $1\frac{1}{2}$ bars. We might call this '3 divided into 2 parts' (hence sharing is called partition). Note: this type of situation may also help children to understand the **equivalence** of $\frac{3}{2}$ and '3 ÷ 2'. The sharing structure often offers a clearer explanation of traditional long division method, particularly if base-ten blocks are used as illustrations (see Calculating 10).

It is important that children learn to distinguish between these two types of situation if their answers to dividing problems are to make sense. If we have a situation where we have some money, for example £32 (the dividend), and we want to know how many tickets each costing £1·50 we can buy with the money, the answer is 21 remainder 50p, not (as a calculator would show) 21·3333 … This is a grouping (or quotition) problem; we want to know how many times £1·50 goes into £32 – there is no sharing involved and fractions make no sense as an answer. This is £1·50 into £32.

On the other hand, if 3 children are going to share 10 fish fingers (the dividend) between them (fairly!) their equal shares will be 3·3333 … ($3\frac{1}{3}$) fish fingers each. In this situation (partition), we already know how many parts the fish fingers are to be divided into (3), but we don't know how big the resulting equal parts will be. This is 10 shared into 3 equal parts.

Unfortunately for children (once again), the language we use when speaking of dividing calculations is often confusing. As you may already have noticed, we tend to use the word 'into' for both sorts of situation – 'dividing 3s into 10' (quotition), as well as 'sharing 10 into 3' (partition). We also often speak of 'dividing 10 by 3'. To complicate things further, when we

introduce mathematical symbols for dividing to children, we often tend to imply different structures for the calculation rather carelessly. For instance, 10 ÷ 3 tends to be read as 'ten divided by three' and is often explained as a 'sharing' problem, whereas $3\overline{)10}$ tends to be read as 'threes into ten', probably because (reading conventionally from left to right) the numbers in this second case appear in the reverse order and we want children to use their tables as they solve it. In both cases we want the children to do the same dividing calculation, but these symbols are often explained as two quite different structures of dividing (the first partition, and the second quotition) and children can very reasonably struggle to understand exactly what it is we want them to do. Do we want them to share 10 into 3 (partition), or to find how many 3s in 10 (quotition)?

Eventually children will come to understand that dividing can be seen either way, but that the way we see a dividing problem will affect the kind of answer we give. Mixing up both structures of dividing without making them distinct to children often leads to confusion, and to answers that don't make sense. In particular, children often struggle to understand what to do with remainders (see page 63).

The **ratio** structure of dividing occurs when something is being scaled down, for example, when a scale model – usually of something large – is made. The classic examples are of course maps, in which large actual geographical distances are all divided by the same number to produce an image that fits onto a piece of paper.

Numicon activities first introduced dividing as grouping (quotition) and left sharing (partition) and fractions resulting from a dividing calculation for subsequent activities. This was done in order to put some mental space between initial experiences of the two structures. Simple **fractions** were, of course, discussed with children earlier, but usually in the context of incomplete units of measure, and not as ways of dividing up whole-number remainders of division calculations.

In Numicon activities, we have always introduced dividing firstly and distinctively as the grouping structure, which emphasizes its **inverse** relation to multiplying. Having shown how 7 × 3 = 21 in multiplying, we then connected this with dividing as grouping (quotition) by asking, 'If 7 times 3 is 21, how many 3s are there in 21?' In Number, Pattern and Calculating 6, we continue to emphasize the **inverse** relationship between dividing and multiplying at every opportunity.

Finally, the **ratio** structure of dividing was first introduced in Number, Pattern and Calculating 1 with halving, and subsequently children have also been invited to find thirds, quarters and sixths of various quantities. This can also be seen as multiplying by a half, a third, a quarter, and so on (again, see **inverse relationships** on page 49), but children are not asked to make this connection with multiplicative inverses explicit yet. Contexts featuring the **ratio** structure of dividing continue to feature in Number, Pattern and Calculating 6 activities.

All dividing structures continue to be relevant to Number, Pattern and Calculating 6 work, especially as children are invited more and more to decide for themselves how to solve problems in context, and thus to decide which arithmetical operations to apply in a situation, and when.

A special note about remainders

When a dividing calculation involving whole numbers doesn't work out exactly, we divide the dividend by the divisor so far as we can, and then find we have a small whole number 'left over'. Children are often confused about what to do with the leftover number; sometimes we leave it as a 'remainder', and sometimes we carry on dividing the remainder into fractions (or decimals). When should we do which? The answer always depends upon the context in which the dividing has arisen, and often the difficulty for children is that there are two quite different reasons for leaving a remainder and other reasons for going into fractions.

The first reason for leaving a remainder is that we are dealing with a quotition situation and fractions would not make sense. If 23 people need taxis home and a taxi will take five people, we divide 5s into 23. The solution to this problem is not that we need $4\frac{3}{5}$ taxis, but that hiring 4 taxis will leave a remainder of 3 people unable to get home (so we'd better order 5 taxis).

The second reason for leaving a remainder is totally different: in a partition situation, we might find ourselves sharing out objects that cannot be broken into smaller parts. If 3 children have 50p to share between them, they can have 16p each and there will be 2p left over that cannot physically be broken down into three equal parts – it therefore has to be left as a remainder.

One reason for continuing to divide a leftover whole number into fractions occurs in a partition situation in which the thing being shared *can* actually be broken down into smaller parts. (The word 'fraction' comes from the same root as the word 'fracture'.) If 6 people are sharing 4 pizzas (4 ÷ 6 as a dividing problem), the solution is not 0 remainder 4 pizzas (everyone gets nothing), but that each person gets $\frac{2}{3}$ (or $\frac{4}{6}$) of a pizza. Note again the usefulness of a situation like this for illustrating the **equivalences** between '4 ÷ 6' and $\frac{4}{6}$ and $\frac{2}{3}$.

A second reason for going into fractions is to do with 'fractions of groups'; in a quotition situation we could say that 18 ÷ 12 = $1\frac{1}{2}$, or 12 goes into 18 'one and a half times'.

Children continue to need much experience with and illustrated discussion of all these kinds of situations in Number, Pattern and Calculating 6.

Methods of multiplying and dividing

As with adding and subtracting, we work on developing children's fluency in multiplying and dividing primarily through ensuring that all work is grounded in a depth of understanding of the natures of these operations, of the types of contexts they are relevant to, and of how (in our base-ten system) both numbers and operations 'fit together'.

As with adding and subtracting, when multiplying and dividing, children should get into the habit of always asking first whether **non-computational thinking** might allow them to change any calculation into an easier (or more convenient) form, and also to estimate what any product or quotient is likely to be, before calculating. For example, 19 × 45 = □ can be thought of as (20 × 45) – 45 = □, and then as 900 – 45 = □, which we would by now expect children to do mentally. Such thinking and estimating is particularly important when dealing with fractions and decimals, to support children in building their understanding of **multiplicative thinking**.

In a scaling down situation, such as modifying a recipe for fewer people, children might need to find a third of 250 g of flour, '$\frac{1}{3}$ × 250 g'. It will probably help them to be able to translate this multiplying calculation into the division 250 ÷ 3, to do the dividing, and then to reason that neither 83·3333 … nor '83 and one left over' are helpful answers in this context, so that 250 g ÷ 3 ≈ 83 g (or even ≈ 80 g) will be the most sensible answer. This is an example of what is called **multiplicative thinking** in context, even though the calculating involved turned out to be dividing.

Children need to learn their multiplication tables, and in the Numicon activities these essential facts have been introduced in an organized sequence, exploiting available patterns along the way to ensure the tables are both connected with each other and memorable. Use of multiplication tables allows children to cope with multiplying small or easy numbers mentally and (as with adding and subtracting) mental methods should always be considered as a first resort. Recalling tables facts quickly is, of course, essential to fluency with all written methods of multiplying and dividing.

Written column methods of short multiplying and dividing were first developed in Number, Pattern and Calculating 4, and were then extended in Number, Pattern and Calculating 5 to include decimal fractions. The 'grid method' of multiplying – crucially built upon experience with visual arrays – was continued as well, and in Number, Pattern and Calculating 5 this was then complemented by the more traditional 'long multiplication' method; both methods in fact reinforce understanding of the **distributive property**, albeit in different ways.

There is a continuing emphasis upon multiplying and dividing by increasingly large multiples of 10 in Number, Pattern and Calculating 6, partly as the easiest examples of scaling up and down to calculate (in our base-ten system), and partly also to support children's extension of whole number multiplying and dividing into multiplying and dividing with decimal fractions. Children are now introduced to column methods of long division, and multiplying and dividing more generally with numbers that can involve calculating up to two decimal places in answers.

In Number, Pattern and Calculating 5, children also began multiplying common fractions, decimals, and mixed numbers by whole numbers. These calculations are often important within measuring contexts involving derived measures, such as finding the paved area of a pathway 7 m × $\frac{1}{2}$m. This work is also importantly connected with multiplying and dividing to find *fractions of* particular amounts, that is, recognizing the **equivalences** between '$\frac{5}{8}$ of 23', '$\frac{5}{8}$ × 23' and '(5 × 23) ÷ 8'.

In Number, Pattern and Calculating 6 children are introduced to multiplying proper fractions together, and although the actual calculating process is (by now) easy for children, making sense of the results generally does depend upon an array illustration (see illustration of $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, page 60). Children are also introduced to dividing proper fractions by whole numbers in Number, Pattern and Calculating 6, e.g. $\frac{1}{6}$ ÷ 2, and there are some interesting possibilities to explore with children about how to do these calculations. One useful avenue to explore here involves investigating the **equivalence** between dividing by 2, and multiplying by $\frac{1}{2}$, and so on.

Multiplicative thinking

There is no doubt that what has come to be called by many people 'multiplicative thinking', or sometimes 'proportional reasoning', is a distinctive way of thinking about relationships that underlies much later work for children, both in mathematics and in science. It would be fair to say that without developing their multiplicative thinking, children will be unable to cope with large swathes of their secondary schooling, or indeed with many crucial aspects of their everyday and working lives. Our physical universe and our social worlds are each filled with multiplicative relationships that we need to be able to manage.

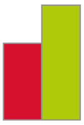
To name a few, sharing, cooking, making drinks, preparing medicines, regulating doses, maps, models, making predictions, assessing risks, measuring speeds, anticipating income, planning spending, engineering, doing science, converting currencies, designing anything, and comparing performances all use multiplicative thinking.

The root idea behind multiplicative thinking is that of a **ratio** comparison (or correspondence), and this is typically contrasted to additive comparisons. There is some evidence that children typically latch on to additive comparisons (He's got more than me! 6 = 3 + 3) significantly before they learn to make multiplicative comparisons (He's got twice as much as me! 6 = 2 × 3), and that ratios and **proportions** are almost universally much harder for us to think about than additive relationships are.

It is important to remember that although this type of thinking is called 'multiplicative', it often involves dividing. This is because multiplying and dividing are essentially 'two sides of the same coin'. As **inverse** operations, each one may 'undo' the other, and almost every dividing problem can be converted into an **equivalent** multiplying problem, for example, $17 \div 6 = 17 \times \frac{1}{6}$. (Dividing by zero cannot be converted into a multiplying problem, which is why we say it has no mathematically valid answer.)

Ratio

A ratio is a *multiplicative* comparison (or correspondence) between two values of the same kind, usually expressed in terms of one of the values being 'x times as much as' the other. Most importantly, the 'number of times' does not have to be a whole number, so comparing a 2 cm length with a 3 cm length multiplicatively, for example, we could say that 3 cm is '$1\frac{1}{2}$ times as long as' 2 cm, or that 2 cm is '$\frac{2}{3}$ as long as' 3 cm. Conventionally, we could write in ratio notation that they are in the ratio 2 : 3 or 3 : 2 to each other, depending on which of them we choose to put first.

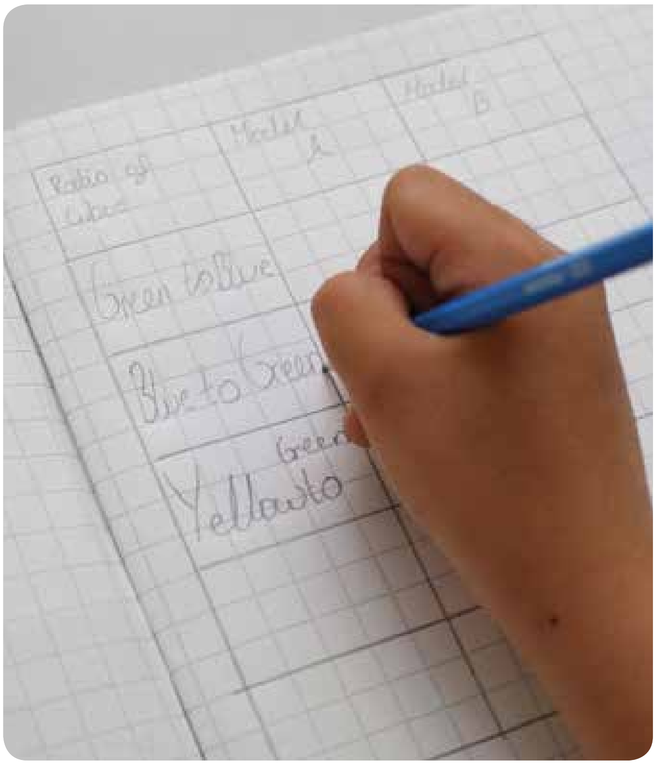


Closely connected with the idea of a ratio is that of a **rate**; a rate relates two measurements of *different* kinds. Speed is a relationship of distance to time, e.g. 30 miles per (or 'for every') hour. Many (but not all) important rates involve time as one of the measurements because time is such a fundamental aspect of our universe, but currency exchange **rates** are often quite important to us as well in everyday life.

Proportion

The term 'proportion' is used in more than one way, unfortunately, but it is perhaps most often used to judge whether two or more sets of values have the same **ratio** to each other within each set. So, in mixing up two separate glasses of orange squash, if they both have the same strength, that is, the same **ratio** of cordial to water as each other, then the two mixtures are said to be **in proportion** (or **proportional**) to each other. This leads to a form in which the idea of **proportion** is often presented, as a relationship between four values:

$$\frac{a}{b} = \frac{c}{d}$$



In words, this might be read as, 'the **ratio** of a to b is the same as the **ratio** of c to d'. In context, this might come out as, 'the **ratio** of cordial to water in this glass is the same as the **ratio** of cordial to water in that glass; the two mixtures are in the same **proportion**'. Problems involving proportions are commonly situations in which three of the four values are known, and the fourth is to be calculated.

(Notice again how fraction notation is sometimes used to specify ratios. This is because there is an **equivalence** between saying, for example, 'these two lengths are in the ratio 2 : 5 to each other', and 'the first length is $\frac{2}{5}$ of the length of the other'. Fractions are often called 'rational numbers'.)

Another use of the word 'proportion' is in contexts where values change in relation to each other; in such situations, a relationship may be either in direct or inverse proportion. In science, mass and weight are said to be directly proportional to each other; this means that if you double the mass of something, you will also double its weight. This directly proportional relationship is what allows us to compare masses in practice by comparing weights.

Scaling up or down is another important context within which proportions have to be directly maintained, for example in recipes, maps, and in geometrical transformations. Doubling a distance on a normal map represents double the distance on the actual ground.

In another context, speed and the time needed for a journey are *inversely* proportional to each other; if you double your speed for a journey, you will halve the amount of time it takes.

Notice how scaling up or down involves multiplying or dividing values by a common factor.

A third use of the word 'proportion' occurs when people ask questions such as, 'What proportion of the tickets were sold to parents?' Two aspects of this use are important: firstly, the words 'proportion' and 'fraction' are interchangeable here; the question could equally well have been, 'What fraction of the tickets …'. And secondly, use of the word 'proportion' usually implies some kind of 'whole' that the proportion is 'of'; ratio comparisons don't imply any 'whole' value, but proportions do.

It is worth highlighting the fact that proportions are often described and expressed in terms of percentages. This is once again because we are essentially concerned with comparisons here, and percentages were designed to make comparisons easy. We would probably respond to the question above by saying, '70% of all the tickets were sold to parents'.

In Number, Pattern and Calculating 6, there is increased attention both to percentages and to ratio and proportion. The contexts in which this work is developed are crucial to children's understanding. Pie charts (see Geometry, Measurement and Statistics 6, Measurement 1) are particularly important illustrations in this context, and the scaling up or down of recipes and maps and models are vital experiences readily related to everyday situations.

Connecting all this together …

You may have noticed, in reading this, just how much all of these ratio comparisons and relationships depend upon children's underlying skills with and understanding of **multiplying** and **dividing**. Since (almost) all dividing can be converted to **equivalent** multiplying, you may now appreciate why all of this work comes to be collected together under the single heading of **multiplicative thinking**.

Multiplying, dividing, ratios, proportion, fractions, decimals, and percentages all link intricately with each other to produce a very varied and complex set of ways to communicate about essentially the same kind of relationships – **multiplicative** relationships. These relationships, and the ways that we communicate about them mathematically, are crucial to children's future progress in both mathematics and in science, as well as in everyday life; they deserve our and our children's full attention. 'Converting' between multiplying and dividing, between common fractions and decimals, between fractions and percentages, between improper fractions and mixed numbers and so on is crucial to success.

Interestingly, in the 2014 National Curriculum[2] a new term appeared with the aim of collecting together a whole range of problem situations that invite **multiplicative thinking**; the term introduced was 'correspondence problems', probably because this work will also underlie the significant later development of those other very important mathematical relationships – **functions** – and functions are essentially about how changes in one variable 'correspond' to changes in another.

In what follows we offer, and conclude with, some explanation and advice on how to connect a range of problems together that will invite children to **think multiplicatively**. When offering these types of problems to children it is important not to suggest or to tell them in advance whether they should multiply, or when to divide; their long-term success depends upon children thinking these things out for themselves.

Correspondence problems

Correspondence problems refer to a variety of situations in which there is said to be a one-to-many correspondence of some kind between two sets of objects. A simple example would be when a number of children are sharing a number of pies; the two sets are the children and the pies. Another example would be when someone has a number of hats and a number of coats and can combine these to make a range of different outfits. In this case, the set of hats 'corresponds' with the set of coats to produce outfits. A third example would be a correspondence between cars and their wheels; each car would correspond to four wheels (usually). Note that, although the correspondence is technically called 'one-to-many', in general the number of elements in each set in a situation can be anything. As a result, such situations are sometimes described as contexts in which n objects

2 See Year 4 in Mathematics programmes of study: key stages 1 and 2 National curriculum in England 2014.

correspond to m objects. In practice, there can be the same number of elements in each set ($n = m$), and which set is specified first (the larger or the smaller) doesn't matter.

The point about correspondence problems is that they usually require **multiplicative thinking**. If one car has four wheels, how many wheels will six cars have? If there are three pies to share equally between twelve children, how much pie will each child get? If a person has five hats and three coats, how many different outfits can they put together? If I have 48 wheels available, how many cars could I make?

Note that, although the thinking involved in all these different situations is called 'multiplicative', some would lead us to multiply and some situations would probably lead us to divide two numbers.

Essentially, a one-to-many correspondence is a way of describing a **ratio**, which could also be expressed as $n:m$ or n to m. Such ratios are at the heart of multiplicative situations. In the sharing case of pies and children, the ratio of pies to children is $3:12$, and each child gets 'three twelfths' of a pie or $\frac{3}{12}$ (written as a fraction).

When we come to hats and coats, there is no sharing going on, but a 'multiplying' of possibilities. For every one of three coats, we can put on one of five different hats, giving 1×5 possibilities for each coat (ratio $1:5$); with three coats the possibilities become $(1 \times 5) + (1 \times 5) + (1 \times 5) = 3 \times 5 = 15$. With wheels and cars, the ratio is $4:1$, so there will always be four times as many wheels as cars.

The thing to remember is that 'correspondence problems' are all essentially to do with **ratio** situations, but we have to think carefully each time whether to multiply or divide to answer a particular question. Later on, as children develop their **multiplicative thinking**, they will come to understand that all dividing can be done by multiplying with fractions, and that there's a reason why fractions are called rational numbers. Make sure that children experience a wide variety of correspondence problems and encourage them to think very carefully about what kind of an answer would make sense in each situation.

Dr Tony Wing – the theory behind Numicon: what we have learned in our work so far

"Teachers using Numicon quickly find themselves learning from children's responses – as do I and the rest of the Numicon authors.

Numicon is a continually growing understanding of the ways in which structured materials and imagery can be helpful to children in their learning of mathematics.

Using Numicon effectively involves understanding something of the theory behind Numicon, including an understanding of what young children face as they learn how to do mathematics.

The following section sets out what I and the other Numicon authors have learned in our work so far, in order to help with the use of the teaching materials."

Doing mathematics – being active and exploring relationships

It is most helpful to see mathematics as an activity, as something people 'do', actively, rather than as a lot of facts and techniques that have to be passively acquired.

The reason mathematics is thought so important for children in school is that we all want them, after they have left school, not just partly to remember those facts and techniques that they used to pass their exams, but to be able to do mathematics successfully when they later meet new and unfamiliar mathematical challenges in their everyday lives and in their work.

Being able to do mathematics involves being able to pick out key relationships in a situation and then manipulating those relationships to predict outcomes that we are interested in.

For example, in a practical shopping situation, key relationships could include those between prices, totals, budgets, currency structures and cash availability. We could manipulate all these to predict whether we can afford to buy something and, if so, how we could pay. Usually, such a situation would require us to do some calculating along the way. Here, the key relationships would be between numbers themselves: we might be manipulating number relationships by adding, or by finding a difference, to predict a total or an amount of change.

It is worth noting three aspects of doing mathematics in the shopping example:

- Firstly, there is the business of working out which quantities are important to us in the given situation, and how they relate to each other.
- Secondly, there is some calculating to do, with pure numbers.
- Thirdly, there is some interpreting of calculation results in order to predict what will happen when we decide what to do in the practical situation.

Identifying and manipulating key relationships in a situation in order to predict outcomes in this way is often called 'mathematical problem solving', and doing this essentially involves **exploring relationships** (the connections between things) within a situation.

Even a problem as simple as finding out, 'How many children are having lunch today?' involves using some kind of order relationship if we are to predict this successfully. For example, not counting the children in order may mean some are missed or some counted twice.

Numicon constantly encourages children to explore the relationships in situations, to see patterns and regularities, and to use these to make predictions. All this lies at the heart of doing mathematics at any level.

It is worth noting that when working on a real-life problem, such as calculating the cost of shopping, we tend to reach a point where we temporarily forget about the practical context we are in and just work with pure 'numbers'. This challenge of moving backwards and forwards between particular practical situations and an abstract world of numbers presents some of the most significant challenges that children face in learning how to do mathematics.

Of course, some problems crop up only within an abstract world of numbers, for example, as we learn how to calculate more effectively. Yet even these situations require us to **be active** and to **explore** and use the various relationships involved between numbers themselves.

Importantly, doing mathematics in our everyday lives and at work involves working out what to do in situations simply as they crop up; there is no helpfully arranged programme to life (as there is in school), and as adults we have to be able to cope with whatever comes up, in whatever order it appears. Children also have to learn to rise to the challenge of being able to do mathematics in new and often unfamiliar situations, not just try to remember selected techniques that are likely to come up at the end of a period studying a particular topic.

This has important implications for both our teaching and our assessing. Children need to learn how to do mathematics in new and unfamiliar situations, how to actively explore relationships and to manipulate relationships between things in the same ways that mathematicians cope with new situations. Children – in their worlds – are learning to join in with the activity of doing mathematics in fresh fields.[1]

1 For more on this view see Freudenthal, H. (1973) *Mathematics as an Educational Task*. Dordrecht: D. Reidel Publishing Company

Generalizing, thinking and communicating

Doing mathematics makes the everyday world predictable in a surprising number of ways. When we board an aircraft, we expect to arrive safely and at our chosen destination; we expect fresh food to be available to us in our local shops as and when we want it; we expect electricity to flow in our homes and in our workplaces whenever we flick on a switch. How is it that these everyday expectations are met so often in most of our lives? The answer is that aeronautical engineers, navigators, logistics experts, electrical engineers and statisticians predict these things for us by doing mathematics.

Crucially, doing mathematics involves a unique way of **thinking** and **communicating** about situations; a special way of communicating that has been developing especially for the purpose of doing mathematics ever since humans first concerned themselves with quantities and relationships.

Interestingly, our mathematical communicating doesn't just happen between us and other people; we also constantly communicate mathematically with ourselves whenever we do mathematics. We call this thinking. Just try multiplying 481 by 37 and listen to the voice in your head as you work it out; you can hear your own thinking as you communicate your calculating with yourself.

It is important to understand how our thinking and our communicating develop. Building on Vygotsky's work, Sfard argues that our thinking develops as our own 'individualized' version of the communicating that we do with others.[2] The important implication is that children's mathematical thinking is their own individual version of the mathematical communicating they do with their teachers. Learning to join in with the mathematical communicating we use around them is how children learn to think and communicate mathematically for, and with, themselves.

2 Sfard, A. (2008) *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. New York: Cambridge University Press

Making our lives predictable with mathematics and the ways that we communicate mathematically are closely connected. In fact, it is because mathematics aims to predict through seeing patterns and regularities in relationships that mathematical thinking and communicating has developed in the distinctive ways that it has. More specifically, mathematics makes situations predictable through **generalizing**. As a result, what we communicate with and about most often when doing mathematics are **generalizations**.

Even the number we call '3' is a generalization; we want children to understand it as meaning '3 of anything'. The number facts children are expected to remember are all generalizations. We want children to understand that '6 + 2 = 8' means '6 of anything and 2 of anything will together always make 8 things – whatever they are'. In geometry, when we talk about the angles of 'a triangle' adding up to 180°, we mean any triangle, not just a particular one we might have drawn in front of us.

Generalizations are important because they can be used to predict outcomes in particular situations. For example, once we've made the generalization that '4 × 25 = 100', we can predict that: the perimeter of a square of side 25 cm will be 100 cm; if we're given £25 a week for four weeks, we'll have been given £100 in all. We could even use it to calculate that:

$$36 \times 25 = (9 \times 4) \times 25 = 9 \times (4 \times 25) = 900$$

In other words, lots of different kinds of particular situations all become much more manageable because of that one generalization.

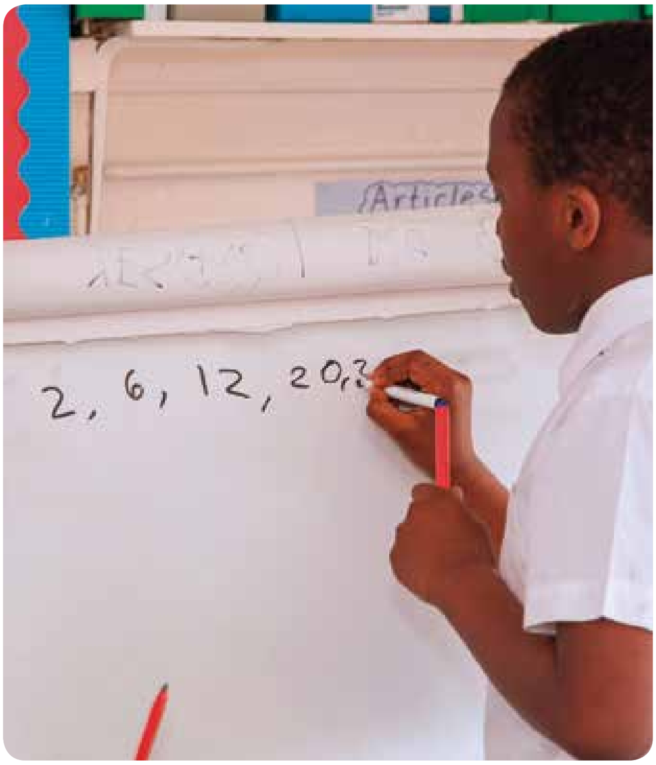
It is important for children to learn 'number facts' such as '6 × 3 = 18', but the most important thing (if such 'facts' are to be useful) is for children to reach them actively through generalizing themselves, and for children to learn how they can use these generalizations in their mathematical thinking and communicating to make the world they live in more predictable.

If 'generalizing' sounds like quite an abstract and difficult thing to do, try thinking of it as simply looking for patterns. The good news is that human beings are all very good at looking for and finding patterns in our experiences. Young children, in particular, have been phenomenally good at this since the day they were born; they learned to speak their mother tongues simply by being extraordinarily attentive to the patterns in the sounds around them – an incredible achievement.

Numicon taps into this incredible facility children have for spotting patterns in situations – for generalizing – wherever possible. This is at the heart of thinking and communicating mathematically.

How do we communicate mathematically?

Communicating with the sheer density of generalizations that we use when we are doing mathematics is not easy.



First of all, this is because generalizations distance us from the close particulars of our individual lives. This is one big reason why so many people find doing mathematics abstract and remote and – mistakenly – feel it is unconnected to their particular everyday world.

Secondly, it is very difficult to talk, and to think, about anything in general without imagining something in particular. When we want to say something about people in general, we usually have experience with particular people in mind (we only ever meet or hear about particular people); when we want to talk with children about triangles in general, we usually show them one triangle (or perhaps a few) in particular. So it is with number generalizations.

When we begin to talk about the generalization '3' with children, we often show them three things in particular (perhaps three counters), even though we want them somehow to interpret those particular three things as representing 3 of anything. 'Seeing the general in the particular'[3] is at the heart of doing mathematics.

The talking that we do in doing mathematics, both with ourselves and with others, is crucial to our developing mathematical thinking and communicating.

Over the centuries, we have developed sophisticated and effective ways of thinking and talking about numbers and other generalizations in mathematics. Not surprisingly, children can sometimes have trouble joining in with them immediately.

3 Mason, J. & Pimm, D. (1984) *Generic Examples: Seeing the General in the Particular, Educational Studies in Mathematics*, 15(3) p277–290

Firstly, in our developed mathematical communicating about amounts and numbers of things in general, we somehow manage to turn our generalizations into objects – mathematical objects. For instance, since it would be very clumsy and awkward to be forever talking about our number generalizations as '6 of anything' and '242 of anything', in practice, we have got used to using a kind of linguistic shorthand and talking about just '6' and '242'. And this has consequences.

In this almost accidental way, we have opted to use number words as nouns and thus to speak and think about the generalizations '6 of anything' and '242 of anything' in the same way as we would if they were actually material objects in the world, like chairs, tables or frogs. We call these generalizations-we-make-into-things, these mathematical objects, numbers. In the previous example, we make the mathematical objects '6' and '242' simply by shortening our generalizing phrases into names.

It is important to realize that by naming something (simply by using a word as a noun), we implicitly announce that what we are talking about is a 'thing', an object; thus just by starting to use number words and symbols as nouns instead of adjectives in our language, we announce to children that in talking about numbers we are talking about things of some kind.

It is also important to remember that, in using number words as nouns, all we do is 'invent' numbers within our use of mathematical language – we don't actually make anything that is real in a world beyond language. A generalization (such as a number) is not a material thing; it is a thought formed in a way of using words (or symbols), and we start talking about such odd things very early in our work with children.

More curiously, most of us who have long ago learned to 'handle' numbers tend to feel as if we literally move them around and connect them as we do calculations, either on paper or 'in our heads' – again as if numbers had the properties of physical things. As numerate adults, we commonly calculate with symbols on a page, or on a calculator display, as if the numerals were somehow the abstract number ideas themselves.

As an example, try dividing 273 by 46 and see if you don't feel as if you are treating the numerals involved as if they are number objects of some kind, and that you just move them around and exchange some for others according to the particular rules you have learned. Don't numbers become just 'things on paper' for you, or numerals in your head or on a calculator display, as you work?[4]

4 Significantly, most of us use imagery of various kinds as well as numerals, as we calculate. But when asked to imagine the number 'ninety four', for example, most of us picture two numerals ('94') – in that order – regardless of whether there is also an image such as a number line involved as well.

As experts, we have come to think and talk about the generalizations we call numbers as things – as objects – as if these mathematical objects were the same kind of things as physical objects that we meet in our everyday worlds. We handle them, we move them around and we line them up on a page. In general, as we do this, we tend to use numerals as if they were the number 'things' that we have invented – simply, remember, by changing how we use number words. How is it possible for children to make sense of all these invisible mathematical 'objects' that we suddenly start talking about in association with numerals in their schooling?

The first thing to observe is that too many children don't ever really make sense of the number objects they meet in their schooling. In practice, we often just expect very young children to move smoothly from talking about 'three sweets' or 'three pencils' (using number words as adjectives referring to physical objects) to talking about just '3' (the same word now used as a noun without any accompanying referents) – as we do.

A little later on, as we introduce fractions, we expect children to move from talking about 'half a pizza', or 'half a bar of chocolate', to talking about just $\frac{1}{2}$ as a thing that is not 'a half of' anything in particular.

This must all seem very strange, and most young children do not so much understand what we are doing as just 'try to go along with it'.

What do we show children as we talk about 'numbers'?

Confusingly for children, at the same point as we start talking about these generalizations (i.e. numbers), using nouns for them as if they were material things, we also commonly start focusing heavily on recording with numerals – as if, by coincidence, the written marks children can actually see on the board or the page are the numbers we are talking about.

There's a good reason for the move to symbols in mathematical communicating at this point: we can only talk about generalizations with what Bruner called symbolic representation.[5] We cannot draw a picture to show '6 of anything' because as soon as we make a picture, or count out 6 physical objects, we are showing children '6 of something'. It is symbolic representation (i.e. words and numerals) in particular that allows us to communicate most readily about invented, non-material things – in this case, pure numbers.

However, unless we are careful, children will quite reasonably – and commonly do (unconsciously) – think that the visible numerals we are showing them actually are the mysterious number things that we have now begun talking about.

The English language is not very helpful either, since, in English, we call numerals 'numbers' as well. We talk about 'the number of that bus' or 'the number of your house' when we are referring to numerals that we can see. In class, we ask children to 'write down' numbers while expecting them to draw numerals.

Importantly, at other times we also want children to understand that '3 and 3 are equal to 6'. What sense can that make to a child who thinks that the numerals '3' and '6' they are looking at are the numbers the teacher is talking about? It is very hard to make sense of such number relationships if all we have to communicate and to think with are number words and symbols; '3' and '3' together don't look like '6' – they look more like '33'.

It is our challenge to find ways of communicating about mathematical objects children can't see (pure numbers) in ways that avoid confusing them with numerals, and which also allow us all to explore relationships between these invisible things.

One key solution is to bring real physical objects and imagery into our communicating; to import special ways of **illustrating**, enactively and visually, relationships between the mystical generalizations we are talking about.

It is for this reason that, when introducing children to numbers, we commonly prepare for and supplement their use of numerals with a range of physical objects and imagery, using actions with objects and visual illustrations to help children 'see' and 'feel' how the invisible number objects we invent with our words relate to each other.[6]

Crucially, this is the point at which we need children to start 'seeing the general' in the particular enactive and visual illustrations that we offer. We can help children a great deal with this in the discussions that we have with them as they work. If we use counters, or cubes, or beads, or beans to talk about numbers, we want children to focus only on 'how many' there are, and to ignore the kinds of physical objects

5 See Bruner, J. (1966) *Towards a Theory of Instruction*. Cambridge, MA: Harvard University Press

6 In this, we are combining Bruner's three modes of representation – enactive, iconic and symbolic – together in order to enrich children's learning as much as possible.

we are using. We need children to stress how many discrete objects are before them, and to ignore what sort of objects they are. 'Stressing and ignoring' lies at the root of 'seeing the general in the particular' illustration, and our conversations with children as we 'illustrate' are crucial.[7] We need to keep asking, 'Would it make any difference to our calculation if those counters were beans? Or what if they were pencils?'

In Bruner's terms, since by thinking and communicating mathematically we are working with generalizations (this is how mathematics helps us to predict), we will eventually do this most effectively by using symbolic representation, e.g. by using numerals and words in our writing and talking to represent our number generalizations.

However, numeral symbols and words are merely conventionally agreed marks on a page and sounds that we hear and – crucially – are dependent for their interpretation upon the prior and accompanying experiences of both action and imagery that led up to the generalizing they symbolize.

Thus, effective use of numerals by children (symbolic representation) in their thinking and communicating – in other words, the calculating we want them to be able to do – is dependent upon their prior and accompanying use of enactive and iconic representation in their experiences with numbers of things.

Children's learning to think and communicate mathematically with generalizations will eventually lead them to mastery of associated symbolic representation (numerals and words). In order to reach and sustain that mastery however, children's route lies necessarily through use of enactive and iconic representation (action and imagery). The most effective teaching therefore involves children 'individualizing' the actions, imagery, words and symbols we use, and joining us in using them in the mathematical communicating of their classrooms.

It is also important to remember that most effective mathematical thinking and communicating at all levels involves a rich blend of enactive, iconic, and symbolic representation together. Action and imagery always support the interpretation of mathematical symbols and there is no point at which children should be expected to leave actions and imagery (physical illustration) behind to do 'grown up' thinking. As children's thinking develops, they 'internalize' the actions and imagery that have led to their effective use of symbols, but physical materials and imagery should always be available in classrooms for children to call upon as new ideas are met, and familiar ones reviewed.

To sum up this part of the theory behind Numicon, numerals (and words) are vitally important symbols that gradually become ciphers for the generalized number objects used in advanced mathematical communicating and thinking.

However, as arbitrary conventional symbols, numerals and words cannot illustrate any number relationships. If children are to learn how to handle number generalizations and their connections effectively, they need to have ways of mediating their mathematical communicating (and thus mathematical thinking) with illustrations that will help children to 'see the general in the particular'.

Sfard (op. cit.) calls the objects and images used for this illustrative purpose 'communication mediators', since they are used to mediate communicating with children. Such objects and imagery subsequently come to mediate children's mathematical communicating with themselves – their thinking about numbers.

Numicon introduces children to thinking and communicating about numbers with a combination of numerals and words in writing and talking, and by mediating this symbolic communicating with physical objects and imagery (enactive and iconic communicating) to illustrate the generalizing that both invents 'numbers' and establishes the relationships between them.

Which communication mediators should we use? Does it matter?

As noted previously, doing mathematics centres around generalizing and using generalizations. In learning how to do mathematics, we become capable of solving an increasing variety of mathematical problems as we call upon an increasing range of generalizations from our past experiences.

Young children usually first learn to generalize about quantities and amounts and talk about number generalizations in their communicating with us, and they then call upon these early generalizations when they later

7 For helpful discussion on this and related views see Mason, J. & Johnston-Wilder, S. (Eds) (2004) *Fundamental Constructs in Mathematics Education*. London: Routledge Falmer. p126ff

need to study, and to calculate with, relationships between quantities and amounts in ever more complex situations.

Children's facility in 'handling' the generalizations we call numbers – their ability to calculate with whole, positive numbers, fractions, negative numbers, for example – is fundamental to almost all of their subsequent progress with mathematics.

We also note that working with numbers – calculating – takes children into a world of invented objects that, although not real, are very significantly connected. Unless children can also begin to generalize about number connections – to generalize about their number generalizations – their calculating will remain very primitive. Typically, children who fail to make much progress with calculating remain restricted to the use of laborious and very basic counting procedures. Fortunately, the regular and systematic relationships between the numbers we invent both makes generalizing about them possible, and also gives us a clue as to which illustrations are most likely to be helpful to children exploring them. Numbers are invented in well-organized systems, and our illustrating needs to reflect their systemic relationships.

In essence, in order to calculate effectively, children need to explore the relationships of numbers to each other. In other words, they need to explore the various ways in which the generalizations of our mathematical communicating about quantities connect with each other. For instance, the ways in which numbers are ordered is important, as are equivalences such as '6 + 2' being equivalent to '8'. The fact that it doesn't matter which way round we add two numbers together, they are still equivalent to the same number total, is important. As is the fact that if we 'add' more than two numbers together it doesn't matter what order we do that in. The numbers '0' and '1' seem to be peculiar in that 'adding 0' and 'multiplying by 1' seem to have no effect at all. Adding seems to be the opposite of subtracting, while multiplying seems to be the opposite of dividing. Interestingly, adding and multiplying seem to be closely connected with each other, as do dividing and subtracting. These are all generalizations about how 'numbers' relate to each other, and they are crucial to the effectiveness of children's calculating.

Scattered (unstructured), random collections of loose objects as illustrations render number relationships, such as those outlined, obscure and are only really useful as opportunities for children to impose relationships upon. Such unorganized collections of, for example, cubes and counters, initially offer opportunities for early counting practice. However, if, as illustrations, they remain unorganized, they are a poor foundation for calculating. It is very difficult to mediate any communicating about number relationships with unorganized collections.

Numicon uses objects and imagery specifically to mediate communicating about number relationships. Numicon brings physical objects and imagery into mathematical communicating that illustrate, above all else, the ways in which numbers are connected. When random collections of discrete objects are introduced in Numicon, children are always expected to put order upon them: to make relationships in what they see.

Numicon also uses a variety of actions, physical objects and imagery because children need to be generalizing from their experiences and conversations and children can only generalize from a variety of experiences.

Number lines of various kinds are used to highlight the order relationships of numbers and Numicon Shapes to give regularity, pattern and two-dimensional shape to number relationships.

Number rods are used to allow children to relate the sizes of numbers to each other in many more ways than are possible with number lines.

Loosely arranged collections of objects are only introduced to offer children important opportunities to impose their own relationships upon such situations: both order structures (when they count) and shape regularities (when they 'find how many' without counting).

By choosing a range of physical materials and imagery suited particularly to illustrating relationships, Numicon offers children the crucial enactive and iconic experiences that enable them to manipulate symbolic representations of their generalizations (numerals and words) effectively in their calculating.

The importance of context

We began this discussion by noting that we want children to leave school able to do mathematics as required in their everyday lives and in their work. In other words, we want children to be able to solve new and unusual mathematical problems when they meet them. Children thus need to be able to explore and pick out key relationships in new and unfamiliar situations, and manipulate them in order to render those situations predictable.

Usually, doing mathematics also involves children moving into the abstract world of number generalizations at key stages – pure calculating – before returning to the practical situation with one or more numbers which need interpreting in the particular context of the problem.

Numicon teaching materials advocate approaching children knowing when the generalized facts and techniques of mathematical communicating are useful through contexts and talking. Within Numicon, each group of activities begins with a carefully chosen particular context, in which the mathematics that is to be learned would be found useful.

Work begins on an activity group by talking about the relationships within which a need for some kind of mathematical response is established; children discuss their initial responses to the questions and challenges involved before moving to the generalizing mathematics to be learned through those activities.

In associated practice activities, children have opportunities to use the general mathematics they are learning in further particular contexts, thus learning from further opportunities to judge when such mathematical generalizing can help.

In Explorer Progress Book tasks, children are presented with challenges that invite them to use mathematics they have been learning, but in unusual and unfamiliar contexts. As their name implies, these tasks invite children to explore relationships in fresh fields.

Summarizing

The key to understanding Numicon is to recognize that doing mathematics involves learning how to **communicate mathematically**, and that mathematical communication is essentially about **generalizations**: it is through generalizing in mathematics that we make our particular worlds predictable.

In the process of doing mathematics, we often make our generalizations into things that we cannot see: *mathematical objects*.

Communicating is at the heart of doing mathematics because that is where our *mathematical objects* – our generalizations – are made.

However, communicating with generalizations is not easy; to help children move between the worlds of mathematical generalizations and of particular situations, we will always need to illustrate our communicating by **being active**, by **illustrating**, and by **talking** as children **explore relationships** they can physically see and feel.

Glossary

Most mathematical terms used in this *Implementation Guide* and the *Number, Pattern and Calculating 6 Teaching Resource Handbook* can be found in a good mathematics dictionary such as the *Oxford Primary Maths Dictionary*.

Other terms you might not be familiar with are explained in this glossary.

base-ten apparatus

A set of concrete materials, systematically designed to help children understand our place value system. Small cubes, sticks of 10 cubes, flat squares of 100 cubes, and large cubes of 1000 small cubes are used when talking about ones, tens, hundreds and thousands respectively. (See Fig. 1.)

bridging across a multiple of 10 when adding or subtracting

Bridging is a calculating technique that involves partitioning (splitting) the number to be added or subtracted. Bridging can be used across any number, but bridging across multiples of 10 is especially useful, since this exploits the very basic adding and subtracting facts to 10 that children learn early on, e.g. 8 + 9 = (8 + 2) + 7 = 17 (see Fig. 2).

Bruner

Jerome Bruner (1915–2016) was an extremely distinguished and influential psychologist. His distinctions between enactive, iconic and symbolic representation have been particularly influential in the field of education.

column value

Numbers are often arranged in columns, with each column having a place value, e.g. hundreds, tens or ones. The numeral '2' in '327' is said to have a column value of two tens. (See also **quantity value**.)

communication mediator

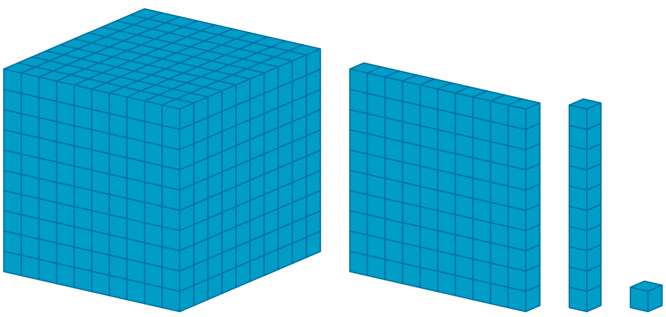
A communication mediator is an object, image or action that helps communication in some way. These communication mediators need to be carefully introduced to children, e.g. Numicon Shapes and number rods become communication mediators when they help to illustrate discussions about the relationships between numbers; number lines and graphs can also act as communication mediators. However, these things only become communication mediators if they are used to support communication. Any physical object or image is just a physical object or image unless it is actually supporting communication; there is 'no magic in the plastic'.

enactive, iconic and symbolic representation

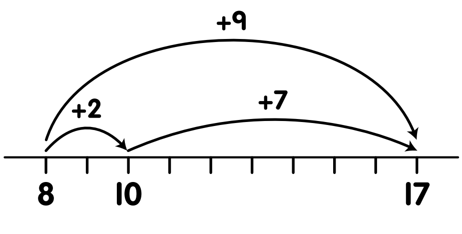
Jerome Bruner (1966) distinguished three key ways in which we humans represent experience to ourselves: through enactive (internalized action), iconic (sensory) and symbolic (forms of language) representations. In Numicon we seek to combine all three forms of representation so that children experience number ideas through action, imagery and conversation.

enumerate

To name how many distinct objects there are in a collection. In Numicon, this term is used in activities that focus on finding how many without counting each individual object; children do this by making Numicon Shape patterns.



1



2

generalization

A statement or observation (not necessarily true) about a whole class of objects, situations, or phenomena. Generalizations are essential and everywhere in mathematics, and for this reason children need to generalize, and to work with generalizations, constantly. Numbers are generalizations, as are rules about numbers, such as, 'it doesn't matter which way around you add two numbers, you will always get the same answer'.

nth term

The nth term rule for a linear sequence of numbers makes it possible to find any term in the sequence without having to find each term in turn. In the sequence 3, 7, 11, 15 … , the nth term is $4n - 1$. So the 15th term will be $4 \times 15 - 1 = 59$.

number fact

The term 'number fact' usually refers to an operation on two or more numbers, together with its outcome. So '6 + 3 = 9' is a 'number fact', as is '256 ÷ 16 = 16'. In UK schools, these are often referred to as 'number bonds'.

number names/objects/words

Adults commonly talk about numbers as if they are objects, i.e. we often use number words such as 'four', or 'twenty-three', as nouns; we ask children questions such as, 'What is seven and three?' In our language, nouns name objects, so we all commonly (and unconsciously) assume that if we use a word as a noun it must be naming an object. So when we use number words as nouns we assume they must be being used to name number objects – thus, according to the way we use words, numbers are often treated as if they are objects.

It is important to remember that we do not always use number words as nouns; quite often we use those same words as adjectives, as in, 'Can you get me three spoons?' One of the key puzzles for children to work out is how (and when) to use number words as adjectives and when as nouns.

number sentence

The metaphor of sentence (from the use of the word 'sentence' in literacy and grammar) is sometimes used to refer to the writing of a number fact in horizontal form, from left to right. So, 4 + 23 = 27 is a number sentence because it is written in the same graphical manner as a normal written sentence in prose.

number trio

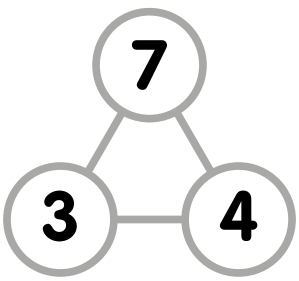
A number trio describes a set of three numbers that relate inverse adding and subtracting facts, e.g. 3, 4 and 7 (see Fig. 3). These are used in Numicon, together with specific forms of illustration, to support children's development of adding and subtracting number facts.

numerals

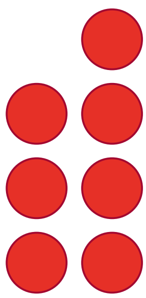
Numerals function as shorthand for number words. The numeral '5' is shorthand for the word 'five'. When used as a noun, the numeral '5' is often said to represent or stand for the number object we call 'five'.

Numicon Shape pattern

When we refer to a Numicon Shape pattern we are referring to the system of arranging objects or images (up to 10 in number) in pairs alongside each other that is sometimes called 'the pair-wise tens frame'. Fig. 4 shows the Numicon 7-pattern.



3


4

Numicon Shape

Numicon Shapes are pieces of coloured plastic with holes (ranging from 1 hole to 10 holes) arranged in the pattern of a pair-wise tens frame (see Fig. 5).

prime factorization

The process of finding the prime numbers that multiply together to give a certain number. The prime factors of 18 are 2, 3 and 3; 2 × 3 × 3 = 18 (see Fig. 6). These prime factors can then be used to find the lowest common multiple and the highest common factor.

Piaget

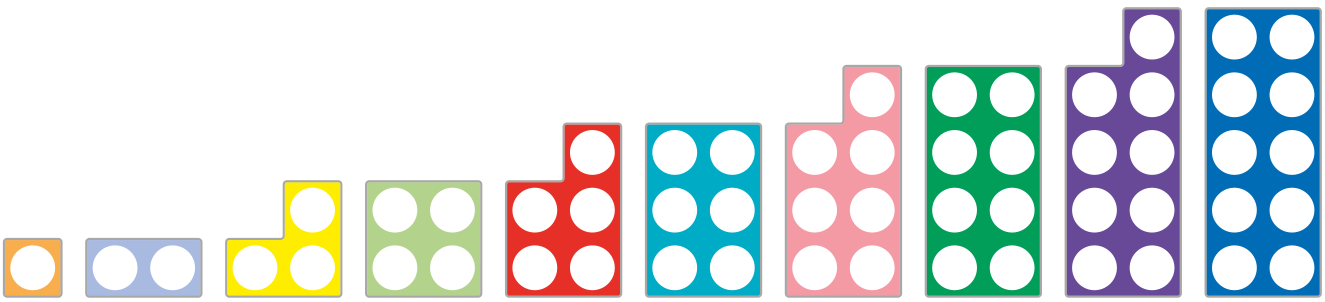
Jean Piaget (1896–1980), was a philosopher and psychologist who spent years studying young children and how they learn. He believed that the development of knowledge is based on developing cognitive structures and that children should be actively involved in their own learning.

quantity value

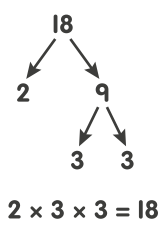
The numeral '2' in '327' is said to have a quantity value of 20 (twenty). (See also **column value**.)

Vygotsky

Lev Vygotsky (1896–1934) contributed a uniquely social dimension to the study of children's thinking. In particular he stressed the role of expert adults in supporting a child's new learning; this occurs optimally in a child's 'zone of proximal development'. Importantly, he saw the development of a child's thinking as crucially influenced by what he characterized as the 'internalization' of speech.



5



6